An Eichler-Shimura isomorphism over function fields between Drinfeld modular forms and cohomology classes of crystals

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Introduction

Quintessential for our understanding of classical modular forms are their various algebraic descriptions. From these one obtains in particular the following properties for the space of cusp forms: The eigenvalues of the Hecke operators, and hence also the coefficients of normalized Hecke eigenforms are algebraic integers. The growth of the absolute values of these eigenvalues is described by the Ramanujan-Petersson conjecture. The Hecke operators satisfy the Eichler-Shimura relations. Finally, one can attach interesting Galois representations to cuspidal Hecke eigenforms.

In analogy to classical modular forms, Goss introduced and studied in [19] so-called Drinfeld modular forms in the realm of function fields. For these many questions which have long been answered in the classical context are still open. An important tool present in the classical situation is the isomorphism of Eichler and Shimura, which allows an interpretation of modular forms as étale cohomology classes. In the function field setting no analogue of this isomorphism is known.

Here we propose to bridge this gap, at least for Drinfeld modular forms of rank two. The novelty in comparison with earlier investigations is the application of the theory of crystals over function fields, developed recently by R. Pink and myself, [4].

In the case at hand, this theory provides a description of Drinfeld modular forms as cohomology classes of crystals over function fields. Crystals in turn have étale realizations and this gives an étale description of Drinfeld modular forms. Using the latter, one can attach Galois representations to cuspidal Hecke eigenforms. Furthermore, the description in terms of crystals should yield a deeper understanding of the Hecke eigenvalues and their growth.

The crystals we obtain should be thought of as a motive over function fields representing a corresponding space of Drinfeld cusp forms. Interpreted this way, the Eichler-Shimura isomorphism says that the analytic realization of the motive is, as a Hecke-module, isomorphic to the space of Drinfeld cusp forms (of a certain weight and level). Therefore one is also interested in the fine structure of these crystals. We show that they are similar to uniformizable t-motives (in the sense of Anderson) and their simple subquotients have real or complex multiplication. The latter is reflected by the fact that the attached Galois representations are all one-dimensional.

Review of the theory of classical modular forms

First we introduce some notation, cf. [31]: For an integer $N \geq 5$, we denote by $Y_1(N) \xrightarrow{g_N} \operatorname{Spec} \mathbb{Z}[1/N]$ the moduli space of elliptic curves over $\mathbb{Z}[1/N]$ with $\Gamma_1(N)$-structure and by $X_1(N) \xrightarrow{\bar{g}_N} \operatorname{Spec} \mathbb{Z}[1/N]$ the compactification from [31]. Let $E_1(N) \xrightarrow{f_N} Y_1(N)$ be the universal elliptic curve with unit section e and relative sheaf of differentials $\omega := e^*\Omega_{E_1(N)/Y_1(N)}$ along the section e. This line bundle can be extended canonically to a line bundle $\bar{\omega}$ on $X_1(N)$, [31] 10.13.9.1. By 'cusps' we denote the cusps of $X_1(N)/\mathbb{C}$ and for $x \in$ cusps we denote by Γ_x the corresponding stabilizer subgroup of $\Gamma_1(N)$. In the following let l be a prime not dividing N.

We define the following objects:

$$S_n^{\mathrm{cl}}(N, \mathbb{C}) := H^0(X_1(N)/\mathbb{C}, \bar{\omega}^{\otimes n}(-\mathrm{cusps})),$$

the space of (classical) holomorphic cusp forms of weight n and level N.

$$H^1_{\mathrm{par},n}(N,\mathbb{Z}) := \mathrm{Ker}\Big(H^1(\Gamma_1(N),\mathrm{Sym}^n\,\mathbb{Z}^2) \to \sum_{x\in\mathrm{cusps}} H^1(\Gamma_x,\mathrm{Sym}^n\,\mathbb{Z}^2)\Big),$$

the parabolic cohomology of $\Gamma_1(N)$ with coefficients in the n-th symmetric power of the tautological representation $\mathbb{Z}^2$ of $\mathrm{GL}_2(\mathbb{Z})$.

$$\tilde{H}^1_{\mathrm{sing},n}(N,\mathbb{Z}) := \mathrm{Im}\Big(H^1_c(X_1(N)/\mathbb{C},\mathrm{Sym}^n\,R^1 f_{N*}\underline{\mathbb{Z}}) \to H^1(X_1(N)/\mathbb{C},\mathrm{Sym}^n\,R^1 f_{N*}\underline{\mathbb{Z}})\Big),$$

where H^1_c and H^1 are the singular cohomology groups with compact support, respectively without supports.

$$\tilde{R}^1_{\text{ét},n}(N,\mathbb{Q}_l) := \mathrm{Im}\Big(R^1_{\text{ét}} g_{N!}\,\mathrm{Sym}^n(R^1_{\text{ét}} f_{N*}\mathbb{Q}_l) \to R^1_{\text{ét}} g_{N*}\,\mathrm{Sym}^n(R^1_{\text{ét}} f_{N*}\mathbb{Q}_l)\Big),$$

where $R^1_{\text{ét}}$ represents the first right derived functor of a morphism between étale topoi. In particular, $\tilde{R}^1_{\text{ét},n}(N,\mathbb{Q}_l)$ is a constructible étale $\mathbb{Q}_l$-sheaf on the scheme $\mathrm{Spec}\,\mathbb{Z}[1/N]$.

On any of the above four objects, one has an action of the Hecke algebra $\mathcal{H}_N$ of correspondences for $\Gamma_1(N)$, [51] Ch. 3.1.

If $\overline{S^{\mathrm{cl}}_n(N,\mathbb{C})}$ denotes the space of antiholomorphic cusp forms of weight n, then there are canonical isomorphisms

$$S^{\mathrm{cl}}_{n+2}(N,\mathbb{C}) \oplus \overline{S^{\mathrm{cl}}_{n+2}(N,\mathbb{C})} \cong H^1_{\mathrm{par},n}(N,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \tilde{H}^1_{\mathrm{sing},n}(N,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C},$$

the so-called Eichler-Shimura isomorphisms, cf. [51, 56]. The comparison of singular and étale topologies yields the isomorphism

$$\tilde{H}^1_{\mathrm{sing},n}(N,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \cong \tilde{R}^1_{\text{ét},n}(N,\mathbb{Q}_l)/\bar{\mathbb{Q}}.$$

All the above isomorphisms are Hecke equivariant with respect to the action of $\mathcal{H}_N$. Finally, for any prime $p \not| lN$, one has the Eichler-Shimura relation for the Hecke operator T_p on the special fiber of $\tilde{R}^1_{\text{ét},n}(N,\mathbb{Q}_l)$ at $\mathrm{Spec}\,\mathbb{F}_p$.

As $H^1_{\mathrm{par},n}(N,\mathbb{Z})$ and $\tilde{H}^1_{\mathrm{sing},n}(N,\mathbb{Z})$ are finitely generated $\mathbb{Z}$-modules, it follows immediately that the Hecke eigenvalues $a_l(f)$ of a cuspidal Hecke eigenform f are algebraic integers, [51] Thm. 3.48. The description via étale cohomology together with Deligne's theorem (the Weil conjectures) gives a proof of the Ramanujan-Petersson conjecture for the growth of the Hecke eigenvalues, [9]. Furthermore, this description yields a continuous Galois representation $\rho_{f,l} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ attached to any Hecke eigenform f and prime l, cf. [9] and [24]. The well-known relation $\mathrm{trace}\rho_{f,l}(\mathrm{Frob}_p) = a_p(f)$ for a prime $p \not| Nl$ is an immediate consequence of the Eichler-Shimura relation.

Drinfeld modular forms

For the analogy between elliptic curves and Drinfeld modules as well as for basics of the theory of Drinfeld modules, we refer to Sections 1 and 4, and [23]. A rather complete treatment of Drinfeld modular forms, will be given in Section 5. We first need to introduce some more notation.

Let C be a complete smooth curve over $\mathbb{F}_q$ and ∞ a chosen closed point on it. Let A be the ring of regular functions on $C \smallsetminus \{\infty\}$ and K its field of fractions. These take on the role of $\mathbb{Z}$ and $\mathbb{Q}$. The completion of K at ∞ is denoted K_∞, and $\mathbb{C}_\infty$ will be the completion of an algebraic closure of K_∞. Furthermore define $\Omega(\mathbb{C}_\infty) := \mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus \mathbb{P}^1(K_\infty)$. This set can be regarded in a natural way as the $\mathbb{C}_\infty$-points of a rigid analytic space Ω over K_∞, cf. [42] or Section 3. We also set $\bar{\Omega}(\mathbb{C}_\infty) := \Omega(\mathbb{C}_\infty) \cup \mathbb{P}^1(K)$. Let $\mathfrak{n}$ be a proper non-zero ideal of A and let $A[1/\mathfrak{n}]$ be the ring of regular functions on $\operatorname{Spec} A \smallsetminus \operatorname{Supp} A/\mathfrak{n}$.

Let $\mathfrak{M}_\mathfrak{n} \xrightarrow{g_\mathfrak{n}} \operatorname{Spec} A[1/\mathfrak{n}]$ denote the moduli space of Drinfeld modules over $A[1/\mathfrak{n}]$ with a full level $\mathfrak{n}$-structure, cf. Definition 1.6. By $\overline{\mathfrak{M}}_\mathfrak{n} \xrightarrow{\bar{g}_\mathfrak{n}} \operatorname{Spec} A[1/\mathfrak{n}]$ its compactification as given by Drinfeld is denoted. The existence of these spaces follows from [10], §8, and [31], Chap. 7. On $\mathfrak{M}_\mathfrak{n}$ there exists a universal Drinfeld module $\varphi_\mathfrak{n}$. The rigidification $\mathfrak{M}_\mathfrak{n}^{\mathrm{rig}}$ of $\mathfrak{M}_\mathfrak{n}/K_\infty$ can be identified with a finite disjoint union $\coprod \Gamma_\nu \backslash \Omega$, where the Γ_ν are suitable arithmetic subgroups of $\mathrm{GL}_2(K)$. The natural compactification of the latter space is the rigidification $\overline{\mathfrak{M}}_\mathfrak{n}^{\mathrm{rig}}$ of $\overline{\mathfrak{M}}_\mathfrak{n}/K_\infty$, which in turn can be described as $\coprod \Gamma_\nu \backslash \bar{\Omega}$. Let $\omega_\mathfrak{n}$ be the line bundle $\coprod \Gamma_\nu \backslash (\mathbb{C}_\infty \times \Omega)$, where an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\nu$ acts on a pair (z, w) by $\gamma(z, w) = (\frac{z}{cw+d}, \frac{aw+b}{cw+d})$. This bundle can be canonically extended to a line bundle $\bar{\omega}_\mathfrak{n}$ on all of $\mathfrak{M}_\mathfrak{n}^{\mathrm{rig}}$, cf. [18], §1, or [15], §6. As in the classical situation, we denote by 'cusps' the cusps of $\overline{\mathfrak{M}}_\mathfrak{n}/\mathbb{C}_\infty$. Corresponding to the decomposition $\mathfrak{M}_\mathfrak{n}^{\mathrm{rig}} \cong \coprod_\nu \Gamma_\nu \backslash \Omega$ we write $\mathrm{cusps} = \coprod \mathrm{cusps}_\nu$. For $x \in \mathrm{cusps}_\nu$, we let Γ_x be the corresponding stabilizer subgroup of Γ_ν. Further discussions may be found in [7] and [42].

Following Goss, [18] §1, one defines the space of Drinfeld cusp forms of weight n and full level $\mathfrak{n}$ as

$$S_n(\mathfrak{n}, \mathbb{C}_\infty) := H^0(\overline{\mathfrak{M}}_\mathfrak{n}^{\mathrm{rig}}, \bar{\omega}_\mathfrak{n}^{\otimes n}(-\mathrm{cusps})).$$

Let Ω_A be the module of differentials of A and define for any $A[\mathrm{GL}_2(K)]$-module M its dual as M^*. Based on the results in [55] by Teitelbaum, one considers the relative group cohomology

$$H^1_{\mathrm{rel},n}(\mathfrak{n}, A, \nu) := H^1(\Gamma_\nu \text{ rel } \{\Gamma_x : x \in \mathrm{cusps}_\nu\}, (\mathrm{Sym}^n \operatorname{Hom}_A(Y_\nu, \Omega_A))^*)$$

for suitable projective A-modules Y_ν of rank 2 and obtains an isomorphism

$$S_{n+2}(\mathfrak{n}, \mathbb{C}_\infty) \cong \bigoplus_\nu H^1_{\mathrm{rel},n}(\mathfrak{n}, A, \nu) \otimes_A \mathbb{C}_\infty.$$

In the function field context, too, one has a natural action of a Hecke algebra on these spaces, [18], §3, or [16]. It will be shown that the above isomorphism is Hecke equivariant. In particular, this shows that the Hecke eigenvalues of the space of cusp forms are integral over A. In Section 6, we will review all of this in detail.

Prior to this work, the above two isomorphism were essentially the only known characterizations of Drinfeld modular forms.

Crystals over function fields

In the following, we want to give a brief account of the theory of crystals over function fields. As a guiding example, one may think of the theory of constructible étale $\mathbb{Q}_l$-sheaves over schemes in characteristic $p \neq l$. However, there are also analogies to the theory of algebraic D-modules. For a morphism $f\colon X \to Y$ of finite type, both of these theories have the six functors f^*, $\otimes$, $f_!$, $f^!$, Hom, f_* postulated by Grothendieck for any good cohomological theory.

The theory of crystals over function fields, constructed by R. Pink and myself, differs from the above two, in that the coefficients as well as the schemes are set in characteristic p. Furthermore, in comparison to the étale theory, the coefficient systems are arbitrary finitely generated $\mathbb{F}_q$-algebras.

The theory of crystals over function fields possesses only the first three of the above six functors. Nevertheless, it is more flexible than Anderson's theory of t-motives, which essentially possesses only the first two of the above functors. Important objects in our theory are given by families of Drinfeld modules or more generally of t-motives. To such, one associates a τ-sheaf, similarly to [53], and the latter represents a crystal.

A further aspect of this theory is its connection with the étale theory of constructible sheaves with characteristic p coefficients. For a coefficient ring B which is a finite $\mathbb{F}_q$-algebra there exists a covariant functor $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{\text{ét}}$ between crystals on X with B-coefficients and constructible étale B-sheaves on X. This functor induces an equivalence of the respective categories which is compatible with the functors $f^*, \otimes, Rf_!$.

As a first application of the theory of crystals over function fields, in [4] Ch. 7, we are able to give an algebraic proof of the rationality of the local L-function of a family of Drinfeld modules, as conjectured by Goss, [53]. The first proof of this, by Taguchi and Wan in [53], was of an analytical nature. Essential for the algebraic proof is a Lefschetz trace formula for crystals, cf. [4]. As a second application one obtains results on special values of Goss' global L-functions at negative integers, as these may be interpreted cohomologically as local L-functions of crystals, [3]. This contributes to a proof of Goss' conjecture that global L-functions over function fields can be meromorphically continued to all of $\mathbb{Z}_p \times \mathbb{C}_\infty^*$, which replaces the complex plane for L-functions over number fields.

Drinfeld modular forms and crystals

Denote by $\underline{\mathcal{F}}_{\mathfrak{n}}$ the crystal attached to the universal Drinfeld module $\varphi_{\mathfrak{n}}$ on $\mathfrak{M}_{\mathfrak{n}}$. This is a family of pure uniformizable A-motives of rank 2. The theory of crystals now suggests, in analogy to the classical situation, the following definition:

$$\underline{\mathcal{S}}_{n+2}(\mathfrak{n}, A) := R^1 g_{\mathfrak{n}!}\Big(\operatorname{Sym}^n \underline{\mathcal{F}}_{\mathfrak{n}}\Big),$$

where $R^1 g_{\mathfrak{n}!}$ is the direct image with compact support functor on crystals for the morphism $g_{\mathfrak{n}}$. This is a crystal on $\operatorname{Spec} A[1/\mathfrak{n}]$ with coefficients in A. As our results described below suggest, the crystal $\underline{\mathcal{S}}_{n+2}(\mathfrak{n}, A)$ should be thought of as a motive over function fields for the cusp forms of weight $n+2$ and level $\mathfrak{n}$.

For a place v of C different from ∞ let $\mathfrak{p}$ denote the corresponding prime ideal of A and A_v the completion of A with respect to v. The inverse system $(\underline{\mathcal{F}}_{\mathfrak{n}} \otimes_A A/\mathfrak{p}^i)^{\text{ét}}_{i\in\mathbb{N}}$ represents a constructible étale A_v-sheaf on $\mathfrak{M}_{\mathfrak{n}}$, which we denote $\underline{\mathcal{F}}_{\mathfrak{n}}^{\text{ét},v}$. We define

$$R^1_{\text{ét}}(n, A_v) := R^1_{\text{ét}} g_{\mathfrak{n}!}\Big(\operatorname{Sym}^n \underline{\mathcal{F}}_{\mathfrak{n}}^{\text{ét},v}\Big),$$

which is a constructible étale A_v-sheaf on $\operatorname{Spec} A[1/\mathfrak{n}]$.

We now describe some of our results:

a) An Eichler-Shimura isomorphism for cusp forms:

In [1], Anderson describes a procedure which attaches to a t-motive M over K_∞ a vector bundle $M\{t\}$ over the rigid space $\operatorname{Spm} K_\infty\langle t\rangle$ together with a $\sigma \times$ id-linear operator τ. The module $M\{t\}$ should be thought of as the analytic motive attached to M and $(M\{t\})^\tau$ as its Betti cohomology. A similar procedure can be applied to the crystal $\underline{\mathcal{S}}_{n+2}(\mathfrak{n}, A)$, cf. Section 9. By $(M\{t\})^\tau$ one denotes the τ-invariants of $M\{t\}$.

Based on Teitelbaum's description of Drinfeld modular forms as harmonic cocycles on the Bruhat-Tits tree, we construct in Section 10 a map

$$\bigoplus_{\nu} H^1_{\mathrm{rel},n}(\mathfrak{n}, A, \nu)^* \longrightarrow \Big(\Big(\underline{\mathcal{S}}_{n+2}(\mathfrak{n}, A)/K_\infty\Big)\{t\}\Big)^{\tau} \tag{1}$$

and show it to be an isomorphism. This should be viewed as an Eichler-Shimura isomorphism. It 'says' that the Betti realization of the 'motive' $\underline{\mathcal{S}}_n(\mathfrak{n}, A)/K_\infty$ after changing coefficients to $\mathbb{C}_\infty$ is dual to the space of cusp forms $S_n(\mathfrak{n}, \mathbb{C}_\infty)$. Note that there is no doubling up as in the classical case.

b) An Eichler-Shimura isomorphism for double cusp forms:

We have a similar construction for the space of double cusp forms: Following Goss, [18], Def. 1.8.3, one defines the space of Drinfeld double cusp forms of weight n as

$$S^2_n(\mathfrak{n}, \mathbb{C}_\infty) := H^0(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathfrak{n}}, \bar{\omega}^{\otimes n}_{\mathfrak{n}}(-2\,\mathrm{cusps})).$$

This is a subspace of $S_n(\mathfrak{n}, \mathbb{C}_\infty)$ of codimension equal to the number of cusps of $\mathfrak{M}_{\mathfrak{n}}$ if $n > 2$ and equal to the number of cusps minus the number of connected components if $n = 2$.

Let $j_{\mathfrak{n}}$ be the open embedding of $\mathfrak{M}_{\mathfrak{n}}$ into $\overline{\mathfrak{M}}_{\mathfrak{n}}$. The extension by zero $j_{\mathfrak{n}!}\underline{\mathcal{F}}_{\mathfrak{n}}$ is the smallest crystal which extends $\underline{\mathcal{F}}_{\mathfrak{n}}$ to $\overline{\mathfrak{M}}_{\mathfrak{n}}$. It turns out that there is another natural extension of $\underline{\mathcal{F}}_{\mathfrak{n}}$ to $\overline{\mathfrak{M}}_{\mathfrak{n}}$, namely the so-called *maximal extension* $\tilde{\underline{\mathcal{F}}}_{\mathfrak{n}} := j_{\mathfrak{n}\#}\underline{\mathcal{F}}_{\mathfrak{n}}$, cf. Section 11. It is based on work of Gardeyn, [13]. In Section 11, we show that for each $n \geq 0$ there is a short exact sequence of crystals on $\overline{\mathfrak{M}}_{\mathfrak{n}}$:

$$0 \longrightarrow j_{\mathfrak{n}!}\,\mathrm{Sym}^n(\underline{\mathcal{F}}_{\mathfrak{n}}) \longrightarrow \mathrm{Sym}^n(\tilde{\underline{\mathcal{F}}}_{\mathfrak{n}}) \longrightarrow \underline{\mathcal{G}}_n \longrightarrow 0.$$

The sheaf $\underline{\mathcal{G}}_n$ can, up to twists with projective A-modules of rank one, be identified with the unit crystal supported on $\overline{\mathfrak{M}}_{\mathfrak{n}} \smallsetminus \mathfrak{M}_{\mathfrak{n}}$. In Section 12, it will be shown that there is a natural isomorphism

$$S^2_n(\mathfrak{n}, \mathbb{C}_\infty)^* \cong \Big(\Big(R^1\bar{g}_{\mathfrak{n}*}(\mathrm{Sym}^n\,\tilde{\underline{\mathcal{F}}}_{\mathfrak{n}})/\mathbb{C}_\infty\Big)\{t\}\Big)^{\tau} \otimes_A \mathbb{C}_\infty. \tag{2}$$

c) $\underline{\mathcal{S}}_n(\mathfrak{n}, A)$ as a Hecke-module:

As in the classical theory, one has an action of Hecke operators on the spaces of Drinfeld cusp forms (and double cusp forms) in their various incarnations: sections of a line bundles, holomorphic functions on $\bar{\Omega}(\mathbb{C}_\infty)$, harmonic cochains, crystals and étale sheaves. We show that all isomorphisms among these are equivariant for the action of the Hecke operators, cf. Sections 6, 10 and 13.

Since the Hecke-algebra is commutative, the above action allows us to decompose $\underline{\mathcal{S}}_n(\mathfrak{n}, A)$ into generalized eigenspaces. Let $M_1, \ldots, M_\lambda$ be torsion free subfactors of $\underline{\mathcal{S}}_n(\mathfrak{n}, A)$, such that the $M_i \otimes_A K$ form a complete list of non-isomorphic simple subfactors of $\underline{\mathcal{S}}_n(\mathfrak{n}, A) \otimes_A K$. The M_i will be shown to be pure uniformizable A_i-motives of rank 1 for a suitable finite extension A_i of A, *and* a suitable characteristic depending on i. We also show that the subfactors of the crystal $R^1g_{\mathfrak{n}*}\underline{\mathcal{G}}_n$ are isomorphic to twists of the unit-motive by finite order characters, which depend on the cusps and can be given explicitly.

d) Galois representations attached to eigenforms:

The above decomposition of $\underline{\mathcal{S}}_n(\mathfrak{n}, A)$ corresponds to decomposing the Hecke-module $S_n(\mathfrak{n}, \mathbb{C}_\infty)$ into generalized eigenspaces. Let f_i denote the normalized eigenform corresponding to M_i.

Using the étale realization of the crystal $\underline{\mathcal{S}}_{n+2}(\mathfrak{n}, A)$ and its compatibility with pushforwards with proper support, one obtains a natural isomorphism

$$(\underline{\mathcal{S}}_{n+2}(\mathfrak{n}, A) \otimes A_v)^{\text{ét}} \cong R^1_{\text{ét}}(n, A_v), \tag{3}$$

which clearly is Hecke-equivariant.

Similar to the classical case, this will enable us to attach 1-dimensional v-adic Galois representations $\rho_{f_i,v}$ to the normalized cuspidal eigenforms f_i for any finite place v of A_i. The Eichler-Shimura relation in characteristic p implies that $T_\mathfrak{p} = \mathrm{Frob}_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ prime to $\mathfrak{n}\mathfrak{p}_v$. In particular, $\rho_{f_i,v}(\mathrm{Frob}_\mathfrak{p}) = a_\mathfrak{p}(f_i)$ in A_{iv}, where $a_\mathfrak{p}(f_i)$ is the eigenvalue of f_i for the Hecke operator $T_\mathfrak{p}$. The results in c) and d) imply that $\rho_{f_i,\nu}$ has finite image whenever f_i is a cuspidal eigenform but not a double cusp form. The simple structure of the crystal $\underline{\mathcal{S}}_0(\mathfrak{n}, A)$ implies the same if f_i is any cuspidal eigenform of weight 2. For doubly cuspidal eigenforms f_i of weight greater than two, the representation ρ_{f_i} will in general have infinite image, as can be seen in examples. This will be explained in detail in Sections 14 and 15.

A survey

Let us now briefly survey the contents of the present work section by section. The first seven sections are largely a review of known material or material that follows very closely the well-known treatment in the case of modular forms. In Section 1, we recall the definitions and main properties of the algebraic moduli spaces $\mathfrak{M}_\mathcal{K}$ of Drinfeld modules with some level-$\mathcal{K}$ structure, e.g. [10], [35]. Section 2 is an expanded version of the discussion of the compactification $\overline{\mathfrak{M}}_\mathcal{K}$ of $\mathfrak{M}_\mathcal{K}$ as given in Drinfeld's seminal paper [10]. We focus mainly on an explicit treatment of the cusps, which seems not to exist in the literature.

As Drinfeld's upper half plane will be important for various explicit computations, we give a fairly complete treatment of it and its quotients by arithmetic subgroups in Section 3. This is first used, in our review of the analytic moduli spaces of Drinfeld modular forms in Section 4. This material mainly stems from [10] and [8].

The following two sections introduce Drinfeld modular forms in their various incarnations and Hecke operators on them. This draws from various sources, e.g., [15, 16, 19, 20, 55]. We give a local as well as an adelic description for all these objects. Certainly none of the adelic parts is available in the literature. Our treatment follows closely that in [52]. Finally, in Section 7, the main facts from the theory of crystals over function fields as developed in [4] are recalled.

The material starting with Section 8 is the core of this article. Before we can come to the first main result of the article, namely a description of Drinfeld modular forms in terms of crystals, some technical issues on families of Drinfeld modules, respectively t-motives have to be resolved.

In Section 8 we develop analytic sites of τ-sheaves and crystals, following closely the theory of algebraic crystals in [4]. An additional difficulty lies in the fact that an extension by zero does not necessarily exist in the analytic context, as one may have essential singularities. Also various compatibilities have to be verified, when passing between different sites.

Then in Section 9, we discuss purity and uniformizability for families. The first is modeled on [41] and the second is an extension of Anderson's pointwise treatment of uniformizability as given in [1]. Having all the above tools available, in Section 10, we state and proof the Eichler-Shimura isomorphism for Drinfeld cusp forms. Uniformizability is a key in our proof. The main result of [40] is also an important auxiliary result.

The proof for double cusp forms is more involved, and uses the concept of *maximal extension* as introduced by Gardeyn in [13]. We further develop this theory in algebraic, analytic and formal contexts in Section 11. Some rather technical computations are needed for certain explicit results on maximal extensions of the crystal corresponding to the universal Drinfeld-module on $\mathfrak{M}_{\mathcal{K}}$. Using these results, we can finally state and proof in Section 12 the Eichler-Shimura isomorphism for double cusp forms.

In Section 13 we define a Hecke action on our crystalline realization of Drinfeld modular forms and show that it is compatible with the usual action under our Eichler-Shimura isomorphism. Also we give a (non-canonical) decomposition of this realization into 'simple' pieces under the Hecke action. An Eichler-Shimura relation on the special fiber of $\mathfrak{M}_{\mathcal{K}}$ over $\mathfrak{p}$ is derived for the Hecke-operator $T_{\mathfrak{p}}$. In the following section, we construct Galois representations from the 'simple' subfactors mentioned above. In this way, we can attach a Galois representation to any cuspidal Drinfeld modular eigenform, where the relation is made precise by using the Eichler-Shimura relation. The final section concludes with an example, where some of the results of this article will be made explicit.

There still remain various open questions for future work.

(i) There should exists a canonically defined subspace of Hecke eigenforms which are cuspidal but not doubly cuspidal. This space needs to be identified. Perhaps one can use Poincaré series to give simple explicit representatives.

(ii) One needs to relate the L-functions attached to the factors of the crystalline realization of Drinfeld modular forms to the L-functions of modular forms as defined by Goss. This would immediatly yield their holomorphy by the results in [3].

(iii) At this point there are no purity results for the pushforward of pure families of t-motives with compact support. As examples show, one cannot expect purity in the same way as for l-adic cohomology, cf. Example 15.5. However it should be possible to give 'bounds on the weights'. (As their is no duality for crystals upper bounds for weights will not imply lower bounds for weights!) This should yield some results on the growth of the eigenvalues of the Hecke operation in the spirit of the Ramanujan-Petersson conjecture. Also the example in Section 15 suggests that the subfactors that arise from Drinfeld cusp forms of fixed weight are pure, however, of verying weights.

(iv) Despite the fact that canonical compactifications for Drinfeld modular schemes of higher rank Drinfeld modules have not been constructed, it does not seem unreasonable to hope to generalize some our results to this setting. For many purposes, such as the analysis of the maximal extension near cusps, it is enough to have a combinatorial description of the codimension one points of a compactification near the cusps, and this might be possible.

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Notation

- p will be a prime, q a power of p and k the field of q elements.
- By X, Y, etc., we denote schemes over k. Their absolute Frobenius endomorphism with respect to k is denoted by σ_X, σ_Y, etc. When it seems redundant, the subscripts are often omitted.
- For a field L we denote by L^{sep} and L^{alg} a separable, respectively algebraic closure.
- We fix a smooth, projective, geometrically connected curve C over k and a closed point ∞ on it.
- The ring of regular functions on $C \smallsetminus \{\infty\}$ is denoted by A, its function field by K and its set of maximal ideals $\mathfrak{p}$ by $\operatorname{Max}(A)$. Because C is geometrically connected, k is the field of constants of K.
- For X, A as above the projection of $X \times \operatorname{Spec} A$ onto the first and second factor are denoted by pr_1 and pr_2, respectively.
- For a non-zero ideal $\mathfrak{n}$ of A, we define its degree $\deg(\mathfrak{n}) := \dim_k A/\mathfrak{n}$. By $V(\mathfrak{n})$ the support of $A/\mathfrak{n}$ is denoted and by $A(\mathfrak{n}) \subset K$ the ring of regular functions on $\operatorname{Spec} A \smallsetminus V(\mathfrak{n})$.
- For any place v of K, we denote by K_v the completion of K at v, by π_v a uniformizing parameter, by A_v its ring of integers and by k_v its residue field. We set $d_v := [k_v : k]$ and $q_v := \operatorname{card}(k_v)$.
- By $|\,.\,|_\infty$ we denote that norm on K_∞ with $|\pi_\infty|_\infty = {q_\infty}^{-1}$, and by $\mathrm{v}_\infty : K_\infty \to \mathbb{Z} \cup \{\infty\}$ the valuation with $\mathrm{v}_\infty(\pi_\infty) = 1$. In particular, $|a|_\infty = \operatorname{card} A/(a) = q^{\deg(a)}$.
- $\mathbb{C}_\infty$ will denote the topological closure of an algebraic closure of K_∞, $\mathcal{O}_{\mathbb{C}_\infty}$ its ring of integers and $\mathfrak{m}_{\mathbb{C}_\infty}$ its maximal ideal. The choice of the algebraic closure of K_∞ provides us with an embedding $\iota_\infty : A \to K_\infty \to \mathbb{C}_\infty$. By $|\,.\,|_{\mathbb{C}_\infty}$ we denote the unique norm on $\mathbb{C}_\infty$ which extends $|\,.\,|_\infty$. Analogously one defines $\mathbb{C}_v$ for any place v of $\operatorname{Spec} A$.
- Say X is a scheme over a ring $R \in \{A, A(\mathfrak{n}), K\}$. Let S be an R-algebra. Then $\mathfrak{b}_S$ is defined as the base change map $X \times_{\operatorname{Spec} R} \operatorname{Spec} S \to X$.
- Define $\hat{A} := \varprojlim A/\mathfrak{n}$, where the limit is taken over all non-zero ideals of A, and let $\mathbb{A}_A^f := \hat{A} \otimes_A K$ be the finite adeles of A. By $\mathbb{A}_A := \mathbb{A}_A^f \times K_\infty$ we denote the adeles of A.
- An A-scheme X will be scheme over A. Similarly, we define A-rings and A-fields. The corresponding morphism $A \to \Gamma(X, \mathcal{O}_X)$ will be denoted by ι_X.
- For any ring R of characteristic p, let $R\{\tau\}$ denote the non-commutative ring of polynomials over R in the indeterminate τ subject to the non-commutation rule $\tau r = r^q \tau$ for $r \in R$. By $\deg_\tau$, we denote the degree of a polynomial in τ.
- For an element $x \in R \otimes_k A$, we denote by $x^{(q^i)}$ the element $(\sigma \times \mathrm{id})^i(x)$. We define this notation also for matrices α over $R \otimes A$ by defining $\alpha^{(q^i)}$ as the matrix obtained by applying $x \mapsto x^{(q^i)}$ to any matrix entry.

1 Drinfeld modular varieties, the algebraic side

In this section we recall the algebraic moduli problem classifying Drinfeld modules of rank r with a level structure. Thereby we set up the notation for various moduli problems we need in the sequel. The main source is [10].

1.1 Definition of a Drinfeld-module

Following [10], we first define a Drinfeld-module over an A-field F and then use this to define it over an arbitrary A-scheme S.

Definition 1.1 *A* Drinfeld-A-module φ on F *is a homomorphism*

$$\varphi\colon A \longrightarrow F\{\tau\} : a \mapsto \varphi_a = \sum_{i\geq 0} \alpha_i(a)\tau^i,$$

satisfying the following two conditions:

(i) $\alpha_0(a) = \iota_F(a)$ *for all* $a \in A$.

(ii) $\varphi_a \neq \iota_F(a)$ *for at least one* $a \in A$.

We often simply speak of Drinfeld-modules instead of Drinfeld-A-modules if the ring A is understood. The kernel of ι_F is called the *characteristic* of φ. If $\ker(\iota_F) = (0)$, then φ is of *generic characteristic.*

Given a Drinfeld-module φ, by [10], Prop. 2.1, there exists a unique positive integer r, called the *rank of* φ, such that $\deg_\tau \varphi_a = r \deg(a)$ for all $a \in A$.

For notational convenience, we let $\mathfrak{G}/S$ be the category of S-group scheme k-vector spaces where S is any A-scheme. Let V be any one-dimensional F-vector space. Viewing V as a torsor over $\mathbb{G}_a/F$, one may identify $F\{\tau\}$ with $\mathrm{End}_{\mathfrak{G}/F}(V)$, which is a more conceptual definition of $F\{\tau\}$. This leads to the following definition, cf. [10].

Definition 1.2 *A* Drinfeld-module $\underline{\varphi} = (\mathcal{L}, \varphi)$ on S of rank r *consists of a line bundle* $\mathcal{L} = \mathcal{L}_\varphi$ *on* S *and a homomorphism*

$$\varphi\colon A \longrightarrow \mathrm{End}_{\mathfrak{G}/S}(\mathcal{L}) : a \mapsto \varphi_a,$$

satisfying the following two properties:

(i) For any A-scheme homomorphism $\operatorname{Spec} F \to S$, *where F is a field, the induced map* $A \longrightarrow \mathrm{End}_{\mathfrak{G}/F}(\mathcal{L}_F)$ *is a Drinfeld-module on F of rank r, where $\mathcal{L}_F$ is the pullback of $\mathcal{L}$ along* $\operatorname{Spec} F \to S$.

(ii) For any a in A, the derivative of φ_a in the tangent space of $\mathcal{L}$ along the zero section is multiplication by $\iota_S(a)$.

A homomorphism $\underline{\xi}\colon \underline{\varphi} \to \underline{\varphi}'$ *of Drinfeld-modules on S, called an* isogeny, *is a surjective* $\xi \in \mathrm{Hom}_{\mathfrak{G}/S}(\mathcal{L}, \mathcal{L}')$ *such that for all $a \in A$ one has* $\xi \circ \varphi_a = \varphi'_a \circ \xi$.

Remark 1.3 Suppose that every line bundle on R is trivial. Then by [10], Prop. 5.2, any Drinfeld module over an A-ring R, is isomorphic to the trivial line bundle together with a homomorphism

$$\varphi\colon A \to R\{\tau\} : a \mapsto \sum_i \alpha_i(a)\tau^i$$

such that $\alpha_0 = \iota_R$, and for all $a \in A$ one has $\deg_\tau \varphi_a = r\deg(a)$ and $\alpha_{r\deg(a)}(a) \in R^*$. This is called the *standard form* of a Drinfeld-module.

A morphism $\underline{\xi}\colon \underline{\varphi} \to \underline{\varphi}'$ of Drinfeld-A-modules in standard form is an element $\xi \in R\{\tau\}$ such that $\xi\varphi_a = \varphi'_a\xi \in R\{\tau\}$ for all $a \in A$. It is an isomorphism precisely when $\xi \in R^* \subset R\{\tau\}$.

1.2 Level structures

We fix an ideal $\mathfrak{n}$ of A, an A-scheme S and a Drinfeld-module $\underline{\varphi}$ on S of rank r.

Definition 1.4 *The* $\mathfrak{n}$-torsion scheme $\underline{\varphi}[\mathfrak{n}]$ of $\underline{\varphi}$ *is the intersection of the kernels of* $\varphi_a : \mathcal{L} \to \mathcal{L}$ *for all* $a \in \mathfrak{n}$.

The scheme $\underline{\varphi}[\mathfrak{n}] \subset \mathcal{L}$ is again in $\mathfrak{G}/S$ and it inherits an action of A which factors through $A/\mathfrak{n}$. A homomorphism between Drinfeld-modules $\underline{\varphi}$ and $\underline{\varphi}'$ induces a homomorphism of S-group schemes $\underline{\varphi}[\mathfrak{n}] \to \underline{\varphi}'[\mathfrak{n}]$.

Suppose $S = \operatorname{Spec} R$, $\mathfrak{n} = (a)$ is principal and $\underline{\varphi}$ is in standard form. For an element $f \in R\{\tau\}$, we denote by $f(x)$ the polynomial in $R[x]$ obtained from f by replacing τ^i by x^{q^i} for all i. Then $\underline{\varphi}[\mathfrak{n}]$ is the scheme $\operatorname{Spec} R[x]/(\varphi_a(x))$. The polynomial $\varphi_a(x)$ is monic of degree $|a|_\infty^r$ with non-zero coefficients only for the terms x^{q^j}, $j = 0, \ldots, r\deg(a)$.

For any abelian group G, by $\underline{G}_S$ we denote the constant group scheme G on S. The following result can be extracted from [10], Prop. 2.2 and 2.3 and the proof of Prop. 5.4:

Proposition 1.5 *The scheme* $\underline{\varphi}[\mathfrak{n}]$ *is finite flat over* S *of rank* $\operatorname{card}(A/\mathfrak{n})^r$. *Its étale locus on* S *is precisely the pullback along* $S \to \operatorname{Spec} A$ *of* $\operatorname{Spec} A(\mathfrak{n})$.

For $T \subset \mathcal{L}$ which is finite flat over S, we denote by $[T]$ the corresponding relative Cartier divisor.

Definition 1.6 *A level* $\mathfrak{n}$*-structure of a Drinfeld-module* $\underline{\varphi}$ *is an* A*-module homomorphism* $\psi : (\mathfrak{n}^{-1}/A)^r \to \underline{\varphi}[\mathfrak{n}](S)$ *such that* $\sum_{g \in (\mathfrak{n}^{-1}/A)^r} [\psi(g)] = [\underline{\varphi}[\mathfrak{n}]]$ *as divisors.*

If the scheme $\underline{\varphi}[\mathfrak{n}]$ is étale over S, then a level $\mathfrak{n}$-structure is simply an isomorphism $\underline{(\mathfrak{n}^{-1}/A)}_S^r \to \underline{\varphi}[\mathfrak{n}]$ in $\mathfrak{G}/S$.

1.3 The moduli problem

For any non-zero ideal $\mathfrak{n}$ of A, one considers the fibered category $\mathcal{M}_\mathfrak{n}^r$ over A-schemes that assigns to each A-scheme S the isomorphism classes of pairs $(\underline{\varphi}, \psi)$, where

(i) $\underline{\varphi}$ is a Drinfeld-module on S of rank r,

(ii) ψ is a level $\mathfrak{n}$-structure of $\underline{\varphi}$.

$\mathcal{M}_\mathfrak{n}^r$ is called the *moduli functor for rank r Drinfeld-modules with level* $\mathfrak{n}$*-structure.*

For a non-zero ideal $\mathfrak{n}$ of A, we define $\mathcal{K}_\mathfrak{n}^r$ as the kernel of the canonical map $\mathrm{GL}_r(\hat{A}) \to \mathrm{GL}_r(A/\mathfrak{n})$. From [10], Cor. on p.577 and from p.578, we obtain:

Theorem 1.7 *Suppose that* $V(\mathfrak{n})$ *contains at least two primes, then* $\mathcal{M}_\mathfrak{n}^r$ *is representable by a regular affine* A*-scheme* $\mathfrak{M}_\mathfrak{n}^r$ *of finite type and dimension* r. *Over* $\operatorname{Spec} A(\mathfrak{n})$, *the scheme* $\mathfrak{M}_\mathfrak{n}^r$ *is smooth of relative dimension* $r-1$.

For $\mathfrak{n}' \subset \mathfrak{n}$, *the induced morphism* $\mathfrak{M}_{\mathfrak{n}'}^r \to \mathfrak{M}_\mathfrak{n}^r$ *is finite and flat. Over* $\operatorname{Spec} A(\mathfrak{n}')$ *it is a Galois cover with Galois group isomorphic to* $\mathcal{K}_\mathfrak{n}^r/\mathcal{K}_{\mathfrak{n}'}^r$.

As we will mainly consider moduli spaces for rank 2 Drinfeld-A-modules, we abbreviate $\mathfrak{M}_\mathfrak{n} := \mathfrak{M}_\mathfrak{n}^2$ and $\mathcal{K}(\mathfrak{n}) := \mathcal{K}_\mathfrak{n}^2$.

1.4 More level structures

We now define further level structures. They will be needed for example when defining Hecke-operators as geometric correspondences.

Let $\mathcal{K}$ be a compact open subgroup of $\mathrm{GL}_r(\hat{A})$. An ideal $\mathfrak{n}$ is called a *conductor* for $\mathfrak{n}$ if $\mathcal{K}(\mathfrak{n}) \subset \mathcal{K}$. Such an ideal always exists and the maximal such ideal is called the *minimal conductor* of $\mathcal{K}$. The action of $\mathcal{K} \subset \mathrm{GL}_r(\hat{A}) \subset \mathrm{GL}_r(\mathbb{A}^f)$ on $(\mathbb{A}^f)^r$ preserves $\hat{A}^r \subset (\mathfrak{n}^{-1}\hat{A})^r$. Therefore $\mathcal{K}$ acts on $(\mathfrak{n}^{-1}/A)^r$, and for any conductor $\mathfrak{n}$, this action on factors via $\mathrm{GL}_r(A/\mathfrak{n})$.

Let $\underline{\varphi}$ be a Drinfeld-module of rank r on an $A(\mathfrak{n})$-scheme S and $\mathfrak{n}$ a conductor of $\mathcal{K}$ as above.

Definition 1.8 *Two level $\mathfrak{n}$-structures $\psi, \psi' : (\mathfrak{n}^{-1}/A)^r \to \underline{\varphi}[\mathfrak{n}](S)$ are called* $\mathcal{K}$-equivalent *if there exists $g \in \mathcal{K}$ such that $\psi' = \psi \circ g$. A* level $\mathcal{K}$-structure $[\psi]$ *of $\underline{\varphi}$ is a $\mathcal{K}$-equivalence class of level $\mathfrak{n}$-structures ψ.*

For an $A(\mathfrak{n})$-scheme S and a Drinfeld-A-module φ on S, denote by $S(\varphi[\mathfrak{n}])$ the étale Galois cover with Galois group $G_{S,\varphi}$ obtained from S by adjoining a complete set of $\mathfrak{n}$-torsion points of φ to S, c.f. Proposition 1.5. The moduli problem of rank r Drinfeld-modules with level $\mathcal{K}$-structure is given by the following fibered category $\mathcal{M}^r_{\mathcal{K}}$ on $A(\mathfrak{n})$-schemes. To an $A(\mathfrak{n})$-scheme S one assigns the set of isomorphism classes of pairs $(\underline{\varphi}, [\psi])$, where

(i) $\underline{\varphi}$ is a Drinfeld-module on S of rank r,

(ii) $[\psi]$ is a level $\mathcal{K}$-structure of the pullback of $\underline{\varphi}$ to $S(\varphi[\mathfrak{n}])$ such that the Galois group $G_{S,\varphi}$ preserves the $\mathcal{K}$-equivalence class of ψ.

Two pairs $(\underline{\varphi}, [\psi])$ and $(\underline{\varphi}', [\psi'])$ are isomorphic if there is an isomorphism $\underline{\xi} : \underline{\varphi} \to \underline{\varphi}'$ on S such that $\underline{\xi}' \circ \psi$ is $\mathcal{K}$-equivalent to ψ', where $\underline{\xi}'$ is the pullback of $\underline{\xi}$ along $S(\varphi[\mathfrak{n}]) \to S$ to $S(\varphi[\mathfrak{n}]) \xrightarrow{\cong} S(\varphi'[\mathfrak{n}])$. To see that $\mathcal{M}^r_{\mathcal{K}}$ defines a fibered category, we need the following lemma:

Lemma 1.9 *Let $\pi\colon S \to S'$ be a morphism of $A(\mathfrak{n})$-schemes, let $(\varphi', [\psi'])$ be in $\mathcal{M}^r_{\mathcal{K}}(S')$ and define $\varphi := \pi^*\varphi'$. Then the following hold:*

(i) There exists a morphism $\tilde{\pi}\colon S(\varphi[\mathfrak{n}]) \to S'(\varphi'[\mathfrak{n}])$ extending π.

(ii) Given any two extensions $\tilde{\pi}_i\colon S(\varphi[\mathfrak{n}]) \to S'(\varphi'[\mathfrak{n}])$, $i = 1, 2$, of π, one has $\tilde{\pi}_1^*[\psi'] = \tilde{\pi}_2^*[\psi']$ and this $\mathcal{K}$-equivalence class is stable under $G_{S,\varphi}$.

Based on the above lemma, one defines in the situation and with the notation of the lemma: $\pi^*[\psi] := \tilde{\pi}^*[\psi']$.

Proof: Suppose first that R and R' are affine and connected, so that we have $S = \operatorname{Spec} R$, $S' = \operatorname{Spec} R'$, $S(\varphi[\mathfrak{n}]) = \operatorname{Spec} \tilde{R}$ and $S'(\varphi'[\mathfrak{n}]) = \operatorname{Spec} \tilde{R}'$ for suitable rings $R, R', \tilde{R}, \tilde{R}'$. Suppose R' is chosen in such a way that the bundle underlying φ' is trivial. This implies that the equations $\varphi'_a(z) = 0$, $a \in \mathfrak{n}$, for the $\mathfrak{n}$-torsion points of φ' are equations in $R'[z]$, and $\tilde{R}'$ is obtained by adjoining a complete set of solutions $\{x'_i\}$ of these equations to R'. The same set of equations (after applying $R' \to R$) has a complete set of solutions $\{x_i\}$ in $\tilde{R}$. By an inductive procedure and possibly relabelling the x_i, one obtains a morphism $\tilde{\pi}\colon \tilde{R}' \to \tilde{R}$ which sends $x'_i \mapsto x_i$, and thus (i) is proved for affine S, S'.

Suppose $\tilde{\pi}_i$, $i = 1, 2$ are lifts of π, where we are still in the affine situation. Let ψ_i denote the level $\mathfrak{n}$-structure for φ on $S(\varphi[\mathfrak{n}])$ induced via $\tilde{\pi}_i$ from ψ. Since the $\tilde{\pi}_i$ are uniquely determined by the images of the elements $\{x'_i\}$, which in turn are zeros of polynomials over R, there exists $\sigma \in G_{S,\varphi}$ such that $\tilde{\pi}_2 = \sigma\tilde{\pi}_1$. Furthermore by considering the situation after specialization to points $x \in S$

and $x' = \pi(x) \in S'$, i.e. the case where all rings involved are fields, there exists $\sigma' \in G_{S',\varphi'}$ such that $\tilde{\pi}_2 = \tilde{\pi}_1\sigma'$ when considered as a map from x to x'. However by the definition of $\tilde{R}'$ and R', the maps $\tilde{\pi}_i$ are uniquely determined by the level structures ψ_i and vice versa, and two level structures on a connected base agree, if they agree at one point. Therefore one also has $\tilde{\pi}_2 = \tilde{\pi}_1\sigma'$.

The second relation between $\tilde{\pi}_1$ and $\tilde{\pi}_2$ shows that ψ_1 and ψ_2 lie in the same $\mathcal{K}$-equivalence class, since by our assumption on ψ' the action of $G_{S',\varphi'}$ preserves the $\mathcal{K}$-equivalence class of ψ'. The first relation now implies that the action of $G_{S,\varphi}$ preserves the $\mathcal{K}$-equivalence class of ψ.

The general case, where S, S' are not necessarily affine, is based on a patching argument. The main point is that a local lift $\tilde{\pi}\colon \operatorname{Spec}\tilde{R} \to \operatorname{Spec}\tilde{R}'$ is uniquely determined by the image $\psi = (\tilde{\pi}')^*\psi'$ of ψ'. Thus if on an affine connected chart one is given maps $\tilde{\pi}_i$, $i = 1, 2$, that agree on a single point, then they agree throughout. Details are left to the reader. ∎

The following lemma, whose basic proof is left as an exercise, compares the above definition with that in the previous subsection and shows its independence of the chosen conductor:

Lemma 1.10 *Suppose that $\mathfrak{n}|\mathfrak{n}'$. Then $\mathcal{M}^r_{\mathcal{K}^r_{\mathfrak{n}'}}$ and $\mathcal{M}^r_{\mathfrak{n}'}$ are naturally isomorphic. In particular the above definition of $\mathcal{M}^r_{\mathcal{K}}$ is independent of the chosen conductor $\mathfrak{n}$.*

Let us first prove a weak representability result for $\mathcal{M}^r_{\mathcal{K}}$:

Proposition 1.11 (relative representability, cf. [31] (4.2)) *Suppose S is a scheme over $A(\mathfrak{n})$ and φ a Drinfeld-A-module on S. Define $\mathcal{M}^r_{\mathcal{K}/(S,\varphi)}$ as the functor from S-schemes to sets defined by mapping $(\pi\colon T \to S)$ to*

$$\{[\psi_T] : [\psi_T] \textit{ is a } G_{T,\pi^*\varphi}\textit{-invariant level-}\mathcal{K} \textit{ structure on } T(\pi^*\varphi[\mathfrak{n}])\}.$$

Then $\mathcal{M}^r_{\mathcal{K}/(S,\varphi)}$ is representable. If S is connected, then it is represented by $\Big(\operatorname{Ind}^{\mathrm{GL}_r(A/\mathfrak{n})}_{G_{S,\varphi}} S(\varphi[\mathfrak{n}])\Big)/\mathcal{K}$.

Proof: For the proof it suffices to assume that S is connected, so that also $S(\varphi[\mathfrak{n}])$ is connected and the automorphism group of level $\mathfrak{n}$-structures is simply $\mathrm{GL}_r(A/\mathfrak{n})$. The action of $G_{S,\varphi}$ on level $\mathfrak{n}$-structures thus identifies $G_{S,\varphi}$ as a subgroup of $\mathrm{GL}_r(A/\mathfrak{n})$. We fix a level $\mathfrak{n}$-structure ψ_0 on $S(\varphi[\mathfrak{n}])$. It defines a level $\mathfrak{n}$-structure $\psi_{\tilde{S}}$ on $\tilde{S} := \operatorname{Ind}^{\mathrm{GL}_r(A/\mathfrak{n})}_{G_{S,\varphi}} S(\varphi[\mathfrak{n}])$. Finally set $S' := \tilde{S}/\mathcal{K}$.

Let now $[\psi_T]$ be a level $\mathcal{K}$-structure on $T(\pi^*\varphi[\mathfrak{n}])$ and ψ_T a representative of it. We may assume that T is connected. By Lemma 1.9, there exists a morphism $\pi' : T(\pi^*\varphi[\mathfrak{n}]) \to S(\varphi[\mathfrak{n}])$ extending π, so that $(\pi')^*\psi_0$ is a level $\mathfrak{n}$-structure of $\pi^*\varphi$ on $T(\pi^*\varphi[\mathfrak{n}])$. Because T is connected there exists $g \in \mathrm{GL}_r(A/\mathfrak{n})$ such that $(\pi')^*(\psi_0 g) = \psi_T$ The definition of $\tilde{S}$ now shows that there is a unique homomorphism $\tilde{\pi} : T(\pi^*\varphi[\mathfrak{n}]) \to \tilde{S}$ extending π such that $\tilde{\pi}^*\psi_{\tilde{S}} = \psi_T$. The same lemma implies that if we chose a different representative of $[\psi_T]$, then one obtains the morphism $g\tilde{\pi}$ for some $g \in \mathcal{K}$. Thus there is a unique morphism $\pi'\colon T \to S'$, independently of the representative of $[\psi_T]$ such that the following diagram commutes:

$$\begin{tikzcd}
T(\pi^*\varphi[\mathfrak{n}]) \arrow[r, "\tilde{\pi}"] \arrow[d] & \tilde{S} \arrow[d] \\
T \arrow[r, "\pi'"] \arrow[dr, "\pi"'] & S' \arrow[d] \\
 & S.
\end{tikzcd}$$

From the definition of π^* on level $\mathcal{K}$-structures, it follows that $(\pi')^*[\psi_{\tilde{S}}] = [\psi_T]$. Since π' is the unique such morphism, the proposition is proved. ∎

To formulate a sufficient condition for the representability of $\mathcal{M}^r_{\mathcal{K}}$, we make the following definition.

Definition 1.12 *A compact-open subgroup $\mathcal{K}$ of $\mathrm{GL}_r(\hat{A})$ is called* admissible *if the intersection of any $\mathrm{GL}_r(\hat{A})$-conjugate of $\mathcal{K}$ with $\mathrm{GL}_r(k) \subset \mathrm{GL}_r(\hat{A})$ is a (possibly trivial) p-group.*

Clearly the above condition is equivalent to the condition that any $\mathrm{GL}_r(\hat{A})$-conjugate of $\mathrm{GL}_r(k)$ intersects $\mathcal{K}$ in a p-group.

Examples are provided by the following simple lemma which is immediate from the injectivity of $\mathrm{GL}_r(k) \hookrightarrow \mathrm{GL}_r(k_v)$ for any place v of C.

Lemma 1.13 *Suppose there exists a place v such that under the canonical map $\mathrm{GL}_r(\hat{A}) \to \mathrm{GL}_r(k_v)$, the image of $\mathcal{K}$ is a p-group. Then $\mathcal{K}$ is admissible. In particular, for $\mathfrak{n}$ a proper non-zero ideal of A the group $\mathcal{K}(\mathfrak{n})$ is admissible.*

The following result clarifies the importance of admissibility:

Lemma 1.14 *If $\mathcal{K}$ is admissible, the action of $\mathcal{K}/\mathcal{K}(\mathfrak{n})$ on $\mathcal{M}^r_{\mathfrak{n}}$ is free.*

Proof: The lemma is proved by contradiction, and so we assume that there exists an $A(\mathfrak{n})$-ring R, an element $g \in \mathcal{K}$, and $(\underline{\varphi}, [\psi]) \in \mathcal{M}^r_{\mathcal{K}}(\operatorname{Spec} R)$ such that $g(\underline{\varphi}, [\psi]) \cong (\underline{\varphi}, [\psi])$. This means that we can find $u \in \mathrm{Aut}(\varphi) \subset R$ such that $(\underline{\varphi}, \psi\bar{g}) = (u\underline{\varphi}u^{-1}, u\psi)$, where $\bar{g}$ is the image of g in $\mathcal{K}/\mathcal{K}(\mathfrak{n})$. We may clearly assume that R is an algebraically closed field, which from now on we will do.

From $\varphi_a = u\varphi_a u^{-1}$ for all non-constant $a \in A$, it follows that $u^{q^{r\deg(a)}-1} = 1$, so that u is a root of unity of order prime to p. The field $\tilde{k} := k[u] \subset R$ defines a finite extension field of k. One now defines a Drinfeld-module over $\tilde{A} := A \otimes_k k'$, which 'extends' φ, namely

$$\tilde{\varphi}\colon \tilde{A} \to R\{\tau\} : \tilde{a} \mapsto \sum \tilde{\alpha}_i(\tilde{a})\tau^i$$

where $\tilde{\varphi}(a \otimes u^i) := \varphi(a)u^i$, and make R into an $\tilde{A}$-module via the map $\tilde{\alpha}_0$ which extends the given ι_R. Because k is the constant field of A, the map $\tilde{\varphi}$ is injective.

Let $\tilde{r}$ be the rank of $\tilde{\varphi}$ so that $r = \tilde{r}s$ with $s = [\tilde{k} : k]$. The module $\varphi[\mathfrak{n}] = \tilde{\varphi}[\mathfrak{n}\tilde{A}]$ is free over $\tilde{A}/\mathfrak{n}\tilde{A} \cong (A/\mathfrak{n}) \otimes_k \tilde{k}$ of rank $\tilde{r}$, because $\tilde{\varphi}$ is injective. If D denotes the companion matrix of u over $M_s(k)$, say with respect to the basis $1, u, \ldots, u^{s-1}$ of $\tilde{k}$ over k, then it follows that u acts on $\varphi[\mathfrak{n}]$ through the matrix

$$\tilde{D} := \begin{pmatrix} D & 0 & \ldots & 0 \\ 0 & D & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & D \end{pmatrix} \in \mathrm{GL}_r(k) \subset \mathrm{GL}_r(A/\mathfrak{n}).$$

Because $u\psi = \psi g$, we have $g'\tilde{D} \in g'g\mathcal{K}(\mathfrak{n})$ for some $g' \in \mathrm{GL}_r(\mathbb{A}^f)$, and hence $\tilde{D} \in g'\mathcal{K}g'^{-1}$. Since $\mathcal{K}$ is admissible, the order of $\tilde{D}$ must be a p-power, and therefore the order of u, which is given by $\tilde{D} \pmod{\mathcal{K}}$ must be a p-power as well. Thus the equation $u^{q^r-1} = 1$ implies $u = 1$, as asserted. ∎

Theorem 1.15 *If $\mathcal{K}$ is admissible of conductor $\mathfrak{n}$, then $\mathcal{M}^r_{\mathcal{K}}$ is representable over* $\operatorname{Spec} A(\mathfrak{n})$ *by an affine $A(\mathfrak{n})$-scheme $\mathfrak{M}^r_{\mathcal{K}}$. The structure morphism $g^r_{\mathcal{K}}$: $\mathfrak{M}^r_{\mathcal{K}} \to \operatorname{Spec} A(\mathfrak{n})$ is smooth of relative dimension $r-1$. The induced morphism*

$$\mathfrak{M}^r_{\mathcal{K}(\mathfrak{n})} \to \mathfrak{M}^r_{\mathcal{K}}$$

is a Galois cover with Galois group isomorphic to $\mathcal{K}/\mathcal{K}(\mathfrak{n})$.

The corresponding universal Drinfeld-module on $\mathfrak{M}^r_{\mathcal{K}}$ is denoted by $\underline{\varphi}^r_{\mathcal{K}}$. For the base change of a moduli scheme to an $A(\mathfrak{n})$-ring R we write $\mathfrak{M}^r_{\mathcal{K},R}$, respectively $\mathfrak{M}^r_{\mathfrak{n},R}$. As in the previous subsection, we always suppress the index r if $r=2$.

Proof: All assertions in the case $\mathcal{K} = \mathcal{K}(\mathfrak{n})$ are shown in [35], Thms. 1.4.1 and 1.5.1. So let now $\mathcal{K}$ be an arbitrary admissible compact-open subgroup of $\mathrm{GL}_r(\hat{A})$ of minimal conductor $\mathfrak{n}$.

Following [31] (4.7), the moduli problem $\mathcal{M}^r_{\mathcal{K}}$ is representable since it is relatively representable by Proposition 1.11 and rigid by Lemma 1.14. The construction in the proof of Proposition 1.11 also shows that $\mathcal{M}^r_{\mathcal{K}}$ is represented by $\mathfrak{M}'_{\mathcal{K}} := (\mathfrak{M}^r_{\mathfrak{n}})/(\mathcal{K}/\mathcal{K}(\mathfrak{n}))$. By [31], Thm. 7.1.3(2), and the previous lemma, it follows that $\mathfrak{M}^r_{\mathfrak{n}} \to \mathfrak{M}'_{\mathcal{K}}$ is an étale $\mathcal{K}/\mathcal{K}(\mathfrak{n})$-torsor. Because $\mathfrak{M}^r_{\mathfrak{n}}$ is smooth over $\operatorname{Spec} A(\mathfrak{n})$ of relative dimension $r-1$, by the case already treated, this shows that $\mathfrak{M}'_{\mathcal{K}}$ has the same properties. ∎

2 Drinfeld's compactification of $\mathfrak{M}_{\mathcal{K}}$.

For lack of a suitably detailed reference, we now reproduce and expand parts of Drinfeld's construction of a regular, respectively smooth compactification $\overline{\mathfrak{M}}_{\mathcal{K}}$ of the moduli schemes $\mathfrak{M}_{\mathcal{K}}$, cf. [10], §9. We will need such an elaborate treatment in Section 12 for our Eichler-Shimura isomorphism for double cusp forms.

The idea of the compactification of the moduli schemes $\mathfrak{M}_{\mathcal{K}}$ is similar to that in the classical situation for arithmetic modular curves, as for example given in [31]. There the Tate curve was used to glue in the missing cusps. The analogue of the Tate curve over functions fields, as realized by Drinfeld, is a family of rank two Drinfeld modules over a discretely valued field which has potential reduction of rank one over the special fiber. The infinitesimal neighborhood of this family along the special fiber serves to describe the degeneration of Drinfeld-modules at the 'cusps'.

This method of compactification by considering degenerations is particularly simple for Drinfeld-modules of rank 2, because over a discretely valued field, any such has potential reduction of rank at least one. If the rank r of the Drinfeld-modules is greater than 2, the degeneration can appear in different ways. So a compactification should contain strata X_s for potential reduction of any rank s between 1 and r such that X_s is in the closure of X_{s+1}. In general, no compactification has been constructed for $\mathfrak{M}^r_{\mathcal{K}}$ for general r and admissible $\mathcal{K}$. For some partial results cf. [14], [29], [39]. Therefore from now on, we only consider the case of rank two.

2.1 A moduli problem

Let $(\varphi, \mathcal{L})$ be a Drinfeld-module on an A-scheme S, so that

$$\varphi\colon A \to \mathrm{End}_{\mathfrak{G}/S}(\mathcal{L}).$$

Define the projective bundle $\overline{\mathcal{L}}$ as the $\mathbb{P}^1$-bundle attached to $\mathcal{L}$ such that $\mathcal{L}$ is the complement in $\overline{\mathcal{L}}$ of the infinite section. The action of A on $\mathcal{L}$ via φ induces an action $\tilde{\varphi}\colon A \smallsetminus \{0\} \to \mathrm{End}(\overline{\mathcal{L}})$ by homogenizing the action of $(\varphi, 1)$ on $\mathcal{L} \oplus \mathcal{O}_S$. (The extendability follows locally on S from the fact that the leading coefficient of φ_a is a unit.) Furthermore, the addition $\mathcal{L} \times \mathcal{L} \xrightarrow{+} \mathcal{L}$ extends uniquely to an action $\mathcal{L} \times \overline{\mathcal{L}} \to \overline{\mathcal{L}}$, again denoted by $+$. In homogeneous local coordinates the latter map is given as the map

$$+\colon \mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 : (x, (y_0 : y_1)) \mapsto (xy_1 + y_0 : y_1).$$

It is compatible with the action given by $\tilde{\varphi}$. In particular via $\tilde{\varphi}$ we have a map of S-points

$$+\colon \mathcal{L}(S) \times \overline{\mathcal{L}}(S) \to \overline{\mathcal{L}}(S) \tag{4}$$

which respects the action of $A \smallsetminus \{0\}$.

Definition 2.1 ([10], § 9) *Fix an admissible subgroup $\mathcal{K}_1$ of $\mathrm{GL}_1(\hat{A})$ and a non-zero ideal $\mathfrak{n}_2$ of A, which may be the unit ideal. Define $\mathcal{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ as* the fibered category of pre-Drinfeld-Tate modules *which assigns to an A-scheme S, the set of triples $(\underline{\varphi}, \psi, \lambda)$, where*

(i) $(\underline{\varphi}, \psi)$ is in $\mathcal{M}_{\mathcal{K}_1}(S)$, and

(ii) $\lambda : \mathfrak{n}_2^{-1} \smallsetminus \{0\} \to \mathrm{Hom}_S(S, \overline{\mathcal{L}})$ is an $A \smallsetminus \{0\}$-homomorphism, i.e., for all $a \in A \smallsetminus \{0\}$ and $m \in \mathfrak{n}_2^{-1} \smallsetminus \{0\}$ one has $\lambda(am) = \tilde{\varphi}_a(\lambda(m))$.

The following result due to Drinfeld motivates the above definition:

Proposition 2.2 *Suppose $\mathbb{V}$ is an A-ring which is a complete discrete valuation ring with fraction field $\mathbb{K}$. Fix an ideal $\mathfrak{n}$ such that $V(\mathfrak{n})$ contains at least two elements. Then there are (natural) bijections between the following sets:*

(i) Isomorphism classes of Drinfeld-modules $\underline{\varphi}$ of rank r over $\mathbb{K}$ with a level $\mathfrak{n}$-structure and reduction of rank $r-1$.

(ii) Isomorphism classes of quadruples $(\underline{\varphi}', \psi', \mathfrak{m}, \lambda)$ where

(i) $\underline{\varphi}'$ is a Drinfeld module $\varphi' : A \to \mathrm{End}_{\mathfrak{G}/\mathbb{V}}(\mathbb{V})$ of rank $r-1$,

(ii) $\psi' : (\mathfrak{n}^{-1}/A)^{r-1} \to \underline{\varphi}'[\mathfrak{n}]$ is a level $\mathfrak{n}$ structure,

(iii) $\mathfrak{m}$ is a fractional ideal of A and

(iv) $\lambda : \mathfrak{n}^{-1}\mathfrak{m} \to \mathbb{K}$ is an A-homomorphism such that the image of λ is discrete in $\mathbb{K}$.

If $r = 2$, then there is a bijection between any of the above sets and:

(iii) Isomorphism classes of Drinfeld-modules $\underline{\varphi}$ of rank 2 over $\mathbb{K}$ with a level $\mathfrak{n}$ structure which do not extend to $\operatorname{Spec}\mathbb{V}$.

Define $G_{\mathbb{K}} := \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$.

Proof: Let $\bar{\mathbb{K}}$ be the separable closure of $\mathbb{K}$ and $\bar{\mathbb{V}}$ its ring of integers. In [10], Prop. 7.2 and 9.1.2, it is shown that for $1 \le r' \le r$ the following sets are in a natural equivalence:

(i) Isomorphism classes of Drinfeld-modules $\underline{\varphi}$ of rank r over $\mathbb{K}$ with potentially semi-stable reduction of rank r'.

(ii) Isomorphism classes of triples $(\underline{\varphi}', \Lambda, \lambda)$ where

(a) $\varphi' : A \to \mathrm{End}_{\mathfrak{G}/\mathbb{V}}(\mathbb{K})$ is a rank r' Drinfeld module with potentially good reduction,

(b) Λ is a projective A-module of rank $r - r'$ and

(c) $\lambda : \Lambda \to \bar{\mathbb{K}} \cup \{\infty\}$ is an A-homomorphism such that the image of Λ is discrete in $\bar{\mathbb{K}}$ and invariant under $G_{\mathbb{K}}$.

The proposition is now a consequence of the following two lemmas. ∎

Lemma 2.3 *Let $\mathbb{V}$ be an A-ring which is a discrete valuation ring with fraction field $\mathbb{K}$. Let $\mathfrak{n}$ be an ideal of A such that $V(\mathfrak{n})$ has cardinality at least 2. Suppose $\varphi : A \to \mathrm{End}_{\mathfrak{G}/\mathbb{K}}(\mathbb{K})$ is a Drinfeld module of rank r with a level $\mathfrak{n}$-structure $\psi : (\mathfrak{n}^{-1}/A)^r \to \varphi[\mathfrak{n}]$. If φ has potentially good reduction, then it has good reduction.*

Proof: The proof rests on the analogue of the criterion of Neron-Ogg-Shafarevich for Drinfeld modules, cf. [54]. To state it, let v be the valuation on $\mathbb{V}$ and $\mathfrak{p} := \{a \in A : \mathrm{v}(\iota_{\mathbb{V}}(a)) \neq 0\}$. For a place v of A, let $\mathrm{Ta}_v(\varphi)$ denote the v-adic Tate-module of φ and $\rho_{\varphi,v} : G_{\mathbb{K}} \to \mathrm{Aut}(\mathrm{Ta}_v(\varphi)) \cong \mathrm{GL}_r(A_v)$ the corresponding Galois representation. Then the criterion says that the following are equivalent:

(i) φ has good reduction.

(ii) For all places v of A such that $\mathfrak{p}_v$ is different from $\mathfrak{p}$, the representation $\rho_{\varphi,v}$ is unramified.

(iii) For some place v of A such that $\mathfrak{p}_v$ is different from $\mathfrak{p}$, the representation $\rho_{\varphi,v}$ is unramified.

Let v be a place of A such that $\mathfrak{p}_v$ is different from $\mathfrak{p}$ and divides $\mathfrak{n}$. Such a v exists as $V(\mathfrak{n})$ contains at least two primes. We claim that $\rho_{\varphi,v}$ is unramified. By the above criterion this will imply the lemma.

As φ has a level $\mathfrak{n}$-structure defined over $\mathbb{K}$, the reduction modulo $\mathfrak{p}_v$ of $\rho_{\varphi,v}$ is trivial. Hence the image of $\rho_{\varphi,v}$ is a pro-p group. In particular, the image of the ramification subgroup $I_{\mathbb{K}}$ of $G_{\mathbb{K}}$ under $\rho_{\varphi,v}$, we write I_ρ for it, is a pro-p group.

As φ has potentially good reduction, by [10], Prop. 7.1, there exists a finite Galois extension $\mathbb{K}'$ of $\mathbb{K}$ of ramification index prime to p with the following property: There exists a Drinfeld-module $\varphi' : A \to \mathrm{End}_{\mathfrak{G}/\mathbb{V}'}(\mathbb{V}')$, where $\mathbb{V}'$ is the ring of integers of $\mathbb{K}'$, such that φ is isomorphic to φ' over $\mathbb{K}'$. By the above Galois criterion this means that the associated v-adic Galois representation $\rho_{\varphi',v} \colon G_{\mathbb{K}'} \to \mathrm{Aut}(\mathrm{Ta}_v(\varphi))$ is unramified. As this representation is isomorphic to the restriction of $\rho_{\varphi,v}$ to $G_{\mathbb{K}'}$, the group I_ρ is a quotient of the inertia subgroup I' of $\mathrm{Gal}(\mathbb{K}'/\mathbb{K})$. As I' is finite and of order prime to p, the pro-p group I_ρ must be trivial which proves that $\rho_{\varphi,v}$ is unramified as asserted. ∎

Lemma 2.4 *Let $\mathfrak{m}$ be a fractional ideal of A and $\lambda : \mathfrak{n}^{-1}\mathfrak{m} \to \mathbb{P}^1(\bar{\mathbb{K}})$ be an $A \setminus \{0\}$-homomorphism such that $\mathrm{Im}\,\lambda$ is discrete in $\bar{\mathbb{K}}$ and invariant under $G_{\mathbb{K}}$. Suppose further that the image of λ carries a level $\mathfrak{n}$-structure, defined over $\mathbb{K}$, i.e., that there is a $G_{\mathbb{K}}$-equivariant isomorphism $\mathfrak{n}^{-1}\,\mathrm{Im}\,\lambda/\,\mathrm{Im}\,\lambda \xrightarrow{\cong} \mathfrak{n}^{-1}/A$, where $\mathfrak{n}^{-1}/A$ is a trivial Galois module. Then $\mathrm{Im}\,\lambda \subset \mathbb{P}^1(\mathbb{K})$.*

Proof: The existence of the level $\mathfrak{n}$-structure implies that the image of $G_{\mathbb{K}}$ in $\mathrm{Aut}(\mathfrak{n}^{-1}\mathfrak{m}) \cong \mathrm{GL}_1(A)$ is a p-group, because its reduction modulo $\mathfrak{n}$ is trivial. But $\mathrm{GL}_1(A) \cong k^*$ is of order prime to p, and hence the image is trivial. ∎

Theorem 2.5 *Suppose $\mathcal{K}_1 \subset \hat{A}^*$ is admissible open and $\mathfrak{n}_2$ is an ideal of A. The fibered category $\mathcal{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ is representable over* $\mathrm{Spec}\,A(\mathfrak{n}_1)$, *where $\mathfrak{n}_1$ is the minimal conductor of $\mathcal{K}_1$. If $V(\mathfrak{n}_1)$ contains at least two elements, it is representable over* $\mathrm{Spec}\,A$.

We write $\mathfrak{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ for the corresponding scheme. In the case $\mathcal{K}_1 = \mathcal{K}^1_{\mathfrak{n}_1}$, we abbreviate $\mathfrak{N}_{\mathfrak{n}_1,\mathfrak{n}_2} := \mathfrak{N}_{\mathcal{K}_1,\mathfrak{n}_2}$.

Proof: Suppose first that $\mathcal{K}_1$ is contained in $\mathcal{K}^1_{\mathfrak{n}'}$ for some $\mathfrak{n}'$ which contains at least two different primes. This implies that the line bundle $\mathcal{L}$ in the universal family of $\mathfrak{M}^1_{\mathcal{K}_1}$ is trivial. Therefore $\overline{\mathcal{L}}$ is isomorphic to $\mathbb{P}^1 \times \mathfrak{M}^1_{\mathcal{K}_1}$, and so if $\mathfrak{n}_2 = (b)$ is principal for some $b \in A$, then $\mathfrak{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ is represented by $\mathbb{P}^1 \times \mathfrak{M}^1_{\mathcal{K}_1}$. Let R denote the coordinate ring of the affine scheme $\mathfrak{M}^1_{\mathcal{K}_1}$, and write $\mathbb{P}^1_R$ for $\mathbb{P}^1 \times \mathrm{Spec}\,R$. Then the universal object is the triple $(\varphi, \psi, \lambda) \in \mathcal{N}_{\mathcal{K}_1,\mathfrak{n}_2}(\mathbb{P}^1_R)$ given by

$$\varphi = \varphi^1_{\mathfrak{n}_1} : A \longrightarrow R\{\tau\} \xrightarrow{\cong} \Gamma(\mathbb{P}^1_R, \mathcal{O}_{\mathbb{P}^1_R})\{\tau\} \cong \mathrm{End}_{\mathfrak{G}/\mathbb{P}^1_R}(\mathbb{G}_{a,\mathbb{P}^1_R})$$

$$a \mapsto \varphi_a := \sum_{i=0}^{\deg(a)} \alpha_i(a)\tau^i,$$

$$\psi = \psi^1_{\mathfrak{n}_1} : (\mathfrak{n}_1^{-1}/A) \xrightarrow{\cong} \varphi[\mathfrak{n}_1] \subset R \xrightarrow{\cong} \Gamma(\mathbb{P}^1_R, \mathcal{O}_{\mathbb{P}^1_R}),$$

and λ is uniquely defined by mapping the generator b^{-1} of $\mathfrak{n}_2^{-1}$ to the first projection

$$\mathrm{pr}_1 \in \mathrm{Hom}(\mathbb{P}^1 \times \mathrm{Spec}\, R, \mathbb{P}^1) \cong \mathrm{Hom}(\mathfrak{M}_{\mathcal{K}_1,\mathfrak{n}_2}, \mathbb{P}^1) \cong \mathrm{Hom}_{\mathfrak{M}_{\mathcal{K}_1,\mathfrak{n}_2}}(\mathfrak{M}_{\mathcal{K}_1,\mathfrak{n}_2}, \mathbb{P}^1_{\mathfrak{M}_{\mathcal{K}_1,\mathfrak{n}_2}}).$$

The images of the other elements of $\mathfrak{n}_2^{-1} \smallsetminus \{0\}$ are uniquely determined by the action of $A \smallsetminus \{0\}$ via $\tilde{\varphi}$.

For general pairs $(\mathcal{K}_1, \mathfrak{n}_2)$ we proceed as in the proof of Theorem 1.15, and apply [31], Thm. 7.1.3, and the following proposition. ∎

To formulate Proposition 2.6 below, we define for any ring R the group

$$B(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R) : c = 0, d = 1 \right\} \cong R^* \ltimes R,$$

and for an open subgroup $\mathcal{K}$ of $\mathrm{GL}_2(\hat{A})$ define $V_{\mathcal{K}}$ as the intersection $\mathcal{K} \cap B(\mathbb{A}^f)$. We abbreviate $V_{\mathfrak{n}} = V_{\mathcal{K}(\mathfrak{n})}$. There is an action of $\gamma = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in B(\hat{A})$ on $(\varphi, \psi, \lambda) \in \mathcal{N}_{\mathcal{K}_1,\mathfrak{n}_2}$, defined by

$$\gamma(\varphi, \psi, \lambda) = (a\varphi, a\psi, \lambda + \psi(b \cdot \_\_)),$$

cf. also page 24.

Proposition 2.6 *Suppose $\mathcal{K}_1 \subset \hat{A}^*$ is admissible open and let $\mathcal{K}_1'$ be an open subgroup of $\mathcal{K}_1$. Suppose further that $\mathfrak{n}_2' \subset \mathfrak{n}_2$ are non-zero ideals of A. Define $\mathfrak{n}$ to be the largest ideal such that $V_{\mathfrak{n}} \subset \mathcal{K}_1' \ltimes (\mathfrak{n}_2'\hat{A})$. Then $\mathcal{N}_{\mathcal{K}_1',\mathfrak{n}_2'}$ is Galois over $\mathcal{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ over $\mathrm{Spec}\, A(\mathfrak{n})$ with Galois group $G = \mathcal{K}_1/\mathcal{K}_1' \ltimes \mathfrak{n}_2/\mathfrak{n}_2'$ which acts freely.*

Furthermore, if $\mathfrak{n}'$ denotes the ideal generated by all the prime divisors of the minimal conductor of $\mathcal{K}_1$, then $\mathcal{N}_{\mathcal{K}_1',\mathfrak{n}_2'}/G \cong \mathcal{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ over $\mathrm{Spec}\, A(\mathfrak{n}')$.

Proof: Each of the groups $V_{\mathfrak{n}}$, $\mathcal{K}_1' \ltimes (\mathfrak{n}_2'\hat{A})$, $\mathcal{K}_1 \ltimes (\mathfrak{n}_2\hat{A})$ is normal in $B(\hat{A})$. This shows that G is well-defined, and that it suffices to prove the lemma in the case $\mathcal{K}_1' \ltimes (\mathfrak{n}_2'\hat{A}) = V_{\mathfrak{n}}$, i.e., from now one we write $\mathcal{K}_1' = \mathcal{K}_{\mathfrak{n}}^1$ and $\mathfrak{n}_2' = \mathfrak{n}$.

Consider the following inclusions of normal subgroups:

$$V_{\mathfrak{n}} \trianglelefteq \mathcal{K}_{\mathfrak{n}}^1 \ltimes (\mathfrak{n}_2\hat{A}) \trianglelefteq \mathcal{K}_1 \ltimes (\mathfrak{n}_2\hat{A}).$$

By Theorem 1.15, the second inclusion clearly corresponds to a Galois cover over $\mathrm{Spec}\, A(\mathfrak{n})$. On the other hand the action of $(\mathcal{K}_1' \ltimes \hat{A})/V_{\mathfrak{n}} \supset (\mathcal{K}_1' \ltimes \mathfrak{n}_2\hat{A})/V_{\mathfrak{n}}$ is fixed point free on $\mathcal{N}_{\mathfrak{n},\mathfrak{n}}$, because

$$(\varphi, \psi, \lambda) \stackrel{!}{=} \gamma(\varphi, \psi, \lambda) = (a\varphi, a\psi, \lambda + \psi(b \cdot \_\_))$$

implies $a - 1 \in \mathfrak{n}\hat{A}$ and $b \in \mathfrak{n}\hat{A}$. The freeness of the action of $(\mathcal{K}_1' \ltimes \hat{A})/V_{\mathfrak{n}}$ also easily implies the second assertion of the Proposition. ∎

Definition 2.7 *Define the affine scheme $\mathfrak{N}^{\infty}_{\mathcal{K}_1,\mathfrak{n}_2}$ as the subscheme $\lambda = \infty$ of $\mathfrak{N}_{\mathcal{K}_1,\mathfrak{n}_2}$, define $\widehat{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$ as the completion of the affine scheme $\mathfrak{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ (as a scheme) along $\mathfrak{N}^{\infty}_{\mathcal{K}_1,\mathfrak{n}_2}$, and define $\widetilde{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$ as the complement of $\mathfrak{N}^{\infty}_{\mathcal{K}_1,\mathfrak{n}_2}$ in $\widehat{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$.*

If $\mathcal{K}_1 = \mathcal{K}^1_{\mathfrak{n}_1}$ for some proper non-zero ideal $\mathfrak{n}_1$ of A, we also write $\mathfrak{n}_1$ instead of $\mathcal{K}_1$ in the notation introduced above.

The definitions of $\widehat{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$, $\widetilde{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$ and $\mathfrak{N}^{\infty}_{\mathcal{K}_1,\mathfrak{n}_2}$ in terms of $\mathfrak{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ directly imply the following result.

Proposition 2.8 *For any $\mathfrak{n}_2$ and admissible $\mathcal{K}_1$, the assertions of Proposition 2.6 hold for $\widehat{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$, $\widetilde{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$, $\mathfrak{N}^{\infty}_{\mathcal{K}_1,\mathfrak{n}_2}$ in place of $\mathfrak{N}_{\mathcal{K}_1,\mathfrak{n}_2}$.*

2.2 The Drinfeld-Tate curve

The scheme $\mathfrak{N}_{\mathcal{K}_1,\mathfrak{n}_2}$ does not carry a Drinfeld-A-module of rank 2 along the generic point of $\mathbb{P}^1$ in any natural way. However its completion $\widetilde{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$ does so. The main purpose of this subsection is to explicitly construct the corresponding Drinfeld-module of rank 2 at least for $\mathcal{K}_1 = \mathcal{K}^1_{\mathfrak{n}_1}$ and $\mathfrak{n}_1$ principal, following the ideas of Proposition 2.2 and [10], § 7. We will assume that $\mathcal{K}_1 = \mathcal{K}^1_{\mathfrak{n}_1}$ for some principal non-zero ideal $\mathfrak{n}_1 = (b)$ of A, as the general case can be obtained from this by Galois descent, cf. Proposition 2.6.

The scheme $\widehat{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$ is the analogue of the Tate curve with level n-structure, $n \in \mathbb{N}$, on $\mathbb{Z}[1/6][[q^{-1/n}]]$ where the first component of the level n-structure is a fixed choice $\mathbb{Z}/(n) \cong \{x \in \bar{\mathbb{Z}} : x^n = 1\}$. We call $\widehat{\mathfrak{N}}_{\mathcal{K}_1,\mathfrak{n}_2}$ the *Drinfeld-Tate module* for $(\mathcal{K}_1, \mathfrak{n}_2)$. In the following we will assume that either $V(\mathfrak{n}_1)$ contains at least two elements, or that we work over $\operatorname{Spec} A(\mathfrak{n}_1)$.

We use the notation from the proof of Theorem 2.5. In particular, R is the coordinate ring of $\mathfrak{M}^1_{\mathcal{K}_1}$ and (φ, ψ, λ) is the universal object in $\mathcal{N}_{\mathfrak{n}_1,\mathfrak{n}_2}(\mathbb{P}^1_R)$. Thus $\widehat{\mathfrak{N}}_{\mathfrak{n}_1,\mathfrak{n}_2} \cong \operatorname{Spec} R[[1/t]]$, and we denote by $(\hat{\varphi}, \hat{\psi}, \hat{\lambda})$ the pullback of (φ, ψ, λ) along the completion map. Thus we have

$$\hat{\varphi} = \varphi^1_{\mathfrak{n}_1} : A \longrightarrow R\{\tau\} \longrightarrow R[[1/t]]\{\tau\} \cong \operatorname{End}_{\mathfrak{G}/\operatorname{Spec} R[[1/t]]}(\mathbb{G}_{a,\operatorname{Spec} R[[1/t]]})$$

$$a \mapsto \hat{\varphi}_a := \sum_{i=0}^{\deg(a)} \alpha_i(a)\tau^i,$$

$$\hat{\psi} : (\mathfrak{n}_1^{-1}/A) \xrightarrow{\cong} \varphi[\mathfrak{n}_1] \subset R \subset R[[1/t]].$$

Finally $\hat{\lambda}$, which arises from the projection $\mathrm{pr}_1 : \mathbb{P}^1 \times \operatorname{Spec} R \to \mathbb{P}^1$ onto the first factor by completing the first scheme at ∞, is given by mapping b^{-1} to

$$(1 : 1/t) \in \mathbb{P}^1(\operatorname{Spec} R[[1/t]]) \cong \operatorname{Hom}_{\operatorname{Spec} R[[1/t]]}(\operatorname{Spec} R[[1/t]], \mathbb{P}^1_{\operatorname{Spec} R[[1/t]]}).$$

In particular, for $a \in A \smallsetminus \{0\}$ one has

$$\hat{\lambda}(ab^{-1}) = (\varphi_a(t) : 1) = \Big(1 : t^{-q^{\deg a}} \frac{1}{\sum_i \alpha_i(a) t^{-q^{\deg a} + q^i}}\Big).$$

The quotient ring $Q(R)$ is a product of fields, and we denote by $\mathbf{K}$ the product of the algebraic closures of these fields. To see that $\widetilde{\mathfrak{N}}_{\mathfrak{n}_1,\mathfrak{n}_2}$ carries a naturally defined Drinfeld-module of rank 2, we will define an exponential map e_λ from $R((1/t)) := R[[1/t]][t]$ to itself: We denote by $\check{\lambda}$ the map

$$\check{\lambda} \colon (b^{-1}) \longrightarrow R((1/t)) : ab^{-1} \mapsto \varphi_a(t),$$

i.e., $\check{\lambda}$ is the first coordinate of $\hat{\lambda}$ if the second coordinate is normalized to 1. Observe that

$$1/\check{\lambda}(ab^{-1}) = \alpha_{\deg(a)}(a)^{-1} \cdot t^{-q^{\deg(a)}} v_a \in R((1/t))^*, \tag{5}$$

for some unit $v_a \in 1 + 1/tR[[1/t]]$, and we have $\alpha_{\deg(a)}(a)^{-1} \in R^*$. This implies that the following expression defines a convergent product in $R[[1/t]][[z]]$:

$$e_\lambda(z) := z \prod_{a \in A \smallsetminus \{0\}} \Big(1 - \frac{z}{\lambda(a)}\Big).$$

Let z be in a finite extension R' of $R((1/t))$. Then all but finitely many terms in the product expansion of $e_\lambda(z)$ are one-units in R'. In particular for any such R' the map $z \mapsto e_\lambda(z) : R' \to R'$ defines a k-linear morphism.

Proposition 2.9 *There exists a unique Drinfeld-module $\varphi' : A \to R((1/t))\{\tau\}$ of rank 2 such that for all $a \in A$ the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & \check{\lambda}(A) \arrow[r] \arrow[d, "\varphi_a"] & R((1/t)) \arrow[r, "e_\lambda"] \arrow[d, "\varphi_a"] & R((1/t)) \arrow[d, "\varphi'_a"] \\
0 \arrow[r] & \check{\lambda}(A) \arrow[r] & R((1/t)) \arrow[r, "e_\lambda"] & R((1/t)).
\end{tikzcd} \tag{6}$$

PROOF: Note first that R is regular and hence reduced because it is the coordinate ring of the moduli space of rank 1 Drinfeld modules with a level $\mathcal{K}_1$-structure. Therefore to prove uniqueness of φ', it suffices to show uniqueness at all fibers with respect to R, i.e. over all rings $R/\mathfrak{m}((1/t))$ where $\mathfrak{m} \in \mathrm{Max}(R)$. Passing to the algebraic closure of $R/\mathfrak{m}((1/t))$, the map e_λ becomes surjective, and hence uniqueness is clear.

For the existence, we define φ'_a by the standard formula of [10], proof of Prop. 3.1: Identify a with $\iota_R(a)$ and let S_a be a set of representatives of $\{x \in \mathbf{K} \smallsetminus \check{\lambda}(A) : \varphi'_a(x) \in \check{\lambda}(A)\}$ modulo $\check{\lambda}(A)$. Define

$$\varphi'_a(z) := az \prod_{c \in S_a} \Big(1 - \frac{z}{e_\lambda(c)}\Big) = z \frac{a}{\prod_{c \in S_a} e_\lambda(c)} \prod_{c \in S_a} \Big(e_\lambda(c) - z\Big).$$

By the following lemma and the invariance of the given expression under the absolute Galois group of $Q(R((1/t)))$, this expression lies indeed in $R((1/t))\{\tau\}$. The proof that

$$\varphi' : A \to R((1/t))\{\tau\} : a \mapsto \varphi'_a$$

does define the structure of a Drinfeld-module of rank 2 on $R((1/t))$ is standard and left to the reader. ∎

Lemma 2.10 *One has*

$$\prod_{c \in S_a} e_\lambda(c) = u_a a$$

for some unit $u_a \in R((1/t))^*$, and the product on the left is independent of the chosen set of representatives.

PROOF: Let $\tilde{S}_a$ be a set of representatives of

$$\{x \in \mathbf{K} : \varphi_a(x) \in \check{\lambda}(A)\} \quad \text{modulo} \quad \check{\lambda}(A) + \varphi[a],$$

which contains 0, i.e. such that the class $\check{\lambda}(A) + \varphi[a]$ is represented by 0. We will assume without loss of generality that $S_a = (\tilde{S}_a + \varphi[a]) \smallsetminus \{0\}$. Let R_a be a set of representatives of $(a^{-1})/A$ which contains 0, and note that $\prod_{c \in \varphi[a]}(x - c) = \varphi_a(c)$ for x an indeterminate and $\varphi_a(\check{\lambda}(\beta)) = \check{\lambda}(a\beta)$ for $\beta \in \mathfrak{n}_2^{-1} \smallsetminus \{0\}$. Then

$$\begin{aligned}
\prod_{c \in S_a} e_\lambda(c) &= \prod_{c \in (\tilde{S}_a + \varphi[a]) \smallsetminus \{0\}} \Big(c \prod_{\alpha \in \check{\lambda}(A) \smallsetminus \{0\}} (1 - \frac{c}{\alpha})\Big) \\
&= \frac{\varphi_a(x)}{x}\Big|_{x=0} \prod_{c \in \tilde{S}_a \smallsetminus \{0\}} \varphi_a(c) \prod_{\alpha \in \check{\lambda}(A) \smallsetminus \{0\}} \frac{\prod_{c \in \tilde{S}_a} \varphi_a(\alpha - c)}{\alpha^{|a|_\infty^2}} \\
&= a \prod_{d \in R_a \smallsetminus \{0\}} \check{\lambda}(d) \prod_{\beta \in A \smallsetminus \{0\}} \frac{\prod_{d \in R_a} \check{\lambda}(a\beta - d)}{\check{\lambda}(\beta)^{|a|_\infty^2}}
\end{aligned}$$

By formula (5) the second factor of the last expression is a unit in $R((1/t))$. Furthermore the numerators and denominators of the third factor are units in $R((1/t))$ and their quotient is a 1-unit for all but finitely many β. Hence the assertion is shown. ∎

2.3 The infinitesimal neighborhood of all cusps

In the previous subsection we explained how to construct what should be the infinitesimal neighborhood of a single cusp. The point of this subsection is to show how to give an adelic description of the infinitesimal neighborhood of the scheme of all cusps of $\mathfrak{M}_{\mathcal{K}}$ for a given admissible $\mathcal{K}$.

By Proposition 2.6, the $\mathfrak{N}_{\mathfrak{n},\mathfrak{n}}$ form naturally an inverse limit system and one defines $\mathfrak{N}$ as $\varprojlim_{\mathfrak{n}} \mathfrak{N}_{\mathfrak{n},\mathfrak{n}}$, and analogously $\widehat{\mathfrak{N}}$ and $\widetilde{\mathfrak{N}}$ — one may take the inverse limit over all principal ideals $\mathfrak{n}$. The scheme $\mathfrak{N}$ represents triples consisting of a rank 1 Drinfeld-module φ on a scheme S, a compatible system of level $\mathfrak{n}$-structures of φ for all non-zero ideals $\mathfrak{n}$, or equivalently an A-homomorphism $\psi\colon K/A \to \mathcal{L}(S)$, and a compatible system of $A\smallsetminus\{0\}$-morphisms $\mathfrak{n}^{-1}\smallsetminus\{0\} \to \overline{\mathcal{L}}$, or equivalently an $A \smallsetminus \{0\}$-homomorphism $\lambda\colon K \smallsetminus \{0\} \to \overline{\mathcal{L}}$. Define $\mathcal{N}$ to be the corresponding moduli problem. Note that the coordinate ring of the affine scheme $\widehat{\mathfrak{N}}$ is simply the integral closure of $R[[1/t]]$ in the infinite Galois extension of $Q(R((1/t)))$ generated by all torsion points of φ'.

There is a natural action of $B(\mathbb{A}^f)$ on $\mathcal{N}$, i.e., on triples (φ, ψ, λ) as above, cf. [10], § 9. To describe it, we first recall the action of $\mathrm{GL}_r(\mathbb{A}^f)$ on $\mathfrak{M}^r = \varprojlim_{\mathfrak{n}} \mathfrak{M}^r_{\mathfrak{n}}$, i.e. the fibered category of pairs $(\tilde{\varphi}, \tilde{\psi})$ consisting of a rank r Drinfeld-module on a scheme S and an A-homomorphism $(K/A)^r \to \mathcal{L}(S)$, [10], § 5:

First consider $g \in \mathrm{GL}_r(\mathbb{A}^f) \cap M_r(\hat{A})$. Then g induces an endomorphism of $(K/A)^r$ with finite kernel H. By [10], Prop. 4.4, there exists a unique isogeny $\tilde{\xi}$ of $\tilde{\varphi}$ to a rank r Drinfeld-module $\tilde{\varphi}'$ whose kernel subgroup scheme is the Cartier divisor $\sum_{\alpha \in H} \tilde{\psi}(\alpha)$. Define $g(\tilde{\varphi}, \tilde{\psi})$ to be the pair $(\tilde{\varphi}', \tilde{\psi}')$ where $\tilde{\psi}'$ is the unique A-homomorphism $(K/A)^r \to \mathcal{L}(S)$ such that the following diagram commutes

$$\begin{tikzcd} (K/A)^r \arrow[r, "\tilde{\psi}"] \arrow[d, "g"'] & \mathcal{L}(S) \arrow[d, "\tilde{\xi}"] \\ (K/A)^r \arrow[r, "\tilde{\psi}'"] & \mathcal{L}(S) \end{tikzcd}$$

where the A-action on the top $\mathcal{L}(S)$ is via $\tilde{\varphi}$ and on the bottom one via $\tilde{\varphi}'$. If g is a scalar matrix for some scalar $a \in A \smallsetminus \{0\}$, then the action of g induces the identity on $\mathfrak{M}^r$. (One simply uses the isogeny $\tilde{\xi} := \tilde{\varphi}_a$ from $\tilde{\varphi}$ to itself.) As $\mathrm{GL}_r(\mathbb{A}^f) = K^*(\mathrm{GL}_r(\mathbb{A}^f) \cap M_2(\hat{A}))$ one can extend the action defined above to all of $\mathrm{GL}_r(\mathbb{A}^f)$ in such a way that K^* acts trivially.

Now one defines the following action of $B(\mathbb{A}^f)$, cf. [10], § 9: For $b \in \mathbb{A}^f$ define

$$b\psi : K \longrightarrow \mathbb{A}^f \xrightarrow{b\cdot} \mathbb{A}^f \longrightarrow \mathbb{A}^f/\hat{A} \cong K/A \xrightarrow{\psi} \mathcal{L}(S).$$

Recall that we have an addition $+ : \mathcal{L}(S) \times \overline{\mathcal{L}}(S) \longrightarrow \overline{\mathcal{L}}(S)$. Thus if for $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in B(\mathbb{A}^f)$ we set

$$g(\varphi, \psi, \lambda) := (a\varphi, a\psi, b\psi + \lambda),$$

this defines a homomorphism $B(\mathbb{A}^f) \to \mathrm{Aut}(\mathfrak{N})$, and it induces actions of $B(\mathbb{A}^f)$ on $\widehat{\mathfrak{N}}$ and $\widetilde{\mathfrak{N}}$. By Proposition 2.6 the scheme $\mathfrak{N}_{\mathfrak{n},\mathfrak{n}}$ is the quotient of $\mathfrak{N}$ by $V_{\mathfrak{n}}$. By Proposition 2.8, the analogous results hold for $\widehat{\mathfrak{N}}$ and $\widetilde{\mathfrak{N}}$.

Define $\widehat{\mathfrak{M}}$ as the scheme $\mathrm{Ind}_{B(\mathbb{A}^f)}^{\mathrm{GL}_2(\mathbb{A}^f)/K^*} \widehat{\mathfrak{N}}$. By this one means the following: The scheme $\widehat{\mathfrak{N}}$ is the affine scheme $\operatorname{Spec} R_\infty$ where $R_\infty = \varinjlim R_{\mathfrak{n}}$ and the $R_{\mathfrak{n}}$ are the coordinate rings $\widehat{\mathfrak{N}}_{\mathfrak{n},\mathfrak{n}}$. The action of $B(\mathbb{A}^f)$ is an action on the ring R_∞. One now considers the set of locally constant functions from $\mathrm{GL}_2(\mathbb{A}^f)/K^*$ to R_∞ which are $B(\mathbb{A}^f)$-equivariant and are non-zero on only finitely many $B(\mathbb{A}^f)$-orbits. (Note that $\mathrm{GL}_2(\mathbb{A}^f)/(K^* B(\mathbb{A}^f))$ is infinite.) The resulting non-unital ring may also be described as $\coprod_{\mathrm{GL}_2(\mathbb{A}^f)/(K^* B(\mathbb{A}^f))} R_\infty$. The spectrum of this latter ring together with the action of $\mathrm{GL}_2(\mathbb{A}^f)/K^*$ is the scheme defined above.

One has the usual formula for the restriction of an induced representation. Say H', H are subgroups of group G and H acts on the affine scheme S from the right. Then

$$\operatorname{Res}_G^{H'} \operatorname{Ind}_H^G S \cong \coprod_{s \in H\backslash G/H'} \operatorname{Ind}_{H'_s}^{H'} S$$

where $H'_s := H' \cap s^{-1}Hs$ and an element $h \in H'_s$ acts on S as $shs^{-1} \in H$.

For an open subgroup $\mathcal{K}$ of $\mathrm{GL}_2(\hat{A})$ one defines $\widehat{\mathfrak{M}}_{\mathcal{K}}$ as $\widehat{\mathfrak{M}}/\mathcal{K}$. Analogously one defines $\widetilde{\mathfrak{M}}_{\mathcal{K}}$ and $\mathfrak{M}_{\mathcal{K}}^{\infty}$, and if $\mathcal{K} = \mathcal{K}(\mathfrak{n})$ we simply write the subscript $\mathfrak{n}$ instead of $\mathcal{K}(\mathfrak{n})$. Noting that K^* acts trivially on $\widehat{\mathfrak{N}}$, the above yields the following explicit expression for $\widehat{\mathfrak{M}}_{\mathcal{K}}$:

Lemma 2.11

$$\widehat{\mathfrak{M}}_{\mathcal{K}} = \coprod_{s \in K^* B(\mathbb{A}^f)\backslash \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}} \widehat{\mathfrak{N}}/(B(\mathbb{A}^f) \cap s^{-1}\mathcal{K}s).$$

If $\mathcal{K} = \mathcal{K}(\mathfrak{n})$, a set of double coset representatives s given as follows: Choose representatives $x_\nu \in (\mathbb{A}^f)^*$ of $\mathrm{Cl}(A) \cong (\mathbb{A}^f)^*/\hat{A}^*$ and let $s_{1,\nu}$ be the diagonal matrix with diagonal entries (x_ν, x_ν). Furthermore choose elements $s_{2,\mu}$ in $\mathrm{GL}_2(\hat{A})$ which form a set of representatives of $k^*\backslash \mathrm{GL}_2(A/\mathfrak{n})/B(A/\mathfrak{n})$. Then the elements $s = s_{1,\nu}s_{2,\mu}$ form a set of representatives for $K^*B(\mathbb{A}^f)\backslash \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$. Because the $s_{1,\nu}$ are in the center of $\mathrm{GL}_2(\mathbb{A}^f)$ and because $\mathcal{K}(\mathfrak{n})$ is normal in $\mathrm{GL}_2(\hat{A})$, one concludes that independently of s one has

$$B(\mathbb{A}^f) \cap s^{-1}\mathcal{K}(\mathfrak{n})s = B(\mathbb{A}^f) \cap \mathcal{K}(\mathfrak{n}) = V_{\mathfrak{n}}.$$

For arbitrary $\mathcal{K}$, the schemes $\widehat{\mathfrak{N}}/(B(\mathbb{A}^f) \cap s^{-1}\mathcal{K}s)$ are called the *formal neighborhoods of the cusps of* $\mathcal{M}_{\mathcal{K}}$. Thus we have shown the following.

Proposition 2.12 *For any $\mathcal{K}$ and any $\mathcal{K}(\mathfrak{n})$ contained in it, one may choose a set of coset representatives of $K^*B(\mathbb{A}^f)\backslash \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ which normalize $\mathcal{K}(\mathfrak{n})$ and $GL_2(\hat{A})$. Furthermore, the formal neighborhoods of all cusps of $\mathfrak{M}_{\mathfrak{n}}$ are isomorphic to $\mathfrak{N}_{\mathfrak{n},\mathfrak{n}}$.*

Remark 2.13 The above proposition needs to be compared with the description of Drinfeld-modules with a level structure, and stable but bad reduction given in Proposition 2.2 (ii). There we needed four parameters to describe such an object, namely in addition to (φ, ψ, λ), we also had an ideal $\mathfrak{m}$ of A. We claim that the fibered category of such triples for a fixed $\mathfrak{m}$ is isomorphic to $\mathcal{N}_{\mathfrak{n},\mathfrak{n}}$, as asserted by the above proposition: To see this choose ν such that the kernel of $x_\nu \colon K/A \to K/A$ is of the form $a\mathfrak{m}^{-1}/A$ for some $a \in K \setminus \mathfrak{m}$. Then multiplication by $s_{1,\nu}$ is an isomorphism between the fibered category for $\mathfrak{m}$ and $\mathcal{N}_{\mathfrak{n},\mathfrak{n}}$.

2.4 Gluing the cusps into $\mathfrak{M}_{\mathcal{K}}$

As the scheme $\widetilde{\mathfrak{M}}_{\mathfrak{n}}$ naturally carries a rank 2 Drinfeld-A-module with a level $\mathfrak{n}$-structure, there is a canonical map $\widetilde{\mathfrak{M}}_{\mathfrak{n}} \to \mathfrak{M}_{\mathfrak{n}}$. (For a modular interpretation due to Drinfeld cf. Proposition 2.2.) This map is used in [10], § 9, to construct a commutative diagram

$$\begin{tikzcd}
\mathfrak{M}_{\mathfrak{n}} \arrow[d] & \widetilde{\mathfrak{M}}_{\mathfrak{n}} \arrow[l] \arrow[r, hook] \arrow[d] & \widehat{\mathfrak{M}}_{\mathfrak{n}} \arrow[d] & \mathfrak{M}_{\mathfrak{n}}^{\infty} \arrow[l, hook'] \arrow[d] \\
\mathbb{A}^1_A & \operatorname{Spec} A((1/t)) \arrow[l] \arrow[r, hook] & \operatorname{Spec} A[[1/t]] & \operatorname{Spec} A, \arrow[l, hook']
\end{tikzcd}$$

which depends on the choice of a non-constant $a \in A$, and where $\operatorname{Spec} A[[1/t]]$ is the infinitesimal thickening of $\infty \in \mathbb{P}^1_A$, all squares are cartesian and all vertical maps are finite flat. To describe the left vertical map in terms of moduli, one assumes, without loss of generality, that all Drinfeld-modules $\underline{\varphi} \in \mathfrak{M}_{\mathfrak{n}}(S)$ are given in standard form

$$\varphi\colon A \mapsto S\{\tau\} : b \mapsto \varphi_b = \sum_{i=0}^{2\deg(b)} \alpha_i(b)\tau^i.$$

Then the left vertical map is defined by

$$\underline{\varphi} \mapsto \Big(\xi_{\underline{\varphi}}\colon A[t] \to S : b \mapsto \iota_S(b), t \mapsto \alpha_{\deg(a)}^{\deg(a)-1}(a)/\alpha_{2\deg(a)}(a)\Big).$$

Details of this construction can be found in the recent work [36].

Clearly the bottom sequence of the above diagram can be glued to yield $\mathbb{P}^1_A$ and the diagram can be used to glue the cusps into $\mathfrak{M}_{\mathfrak{n}}$ and hence to construct a finite scheme $\widehat{\mathfrak{M}}_{\mathfrak{n}}$ together with a finite flat morphism to $\mathbb{P}^1_A$.

Theorem 2.14 *Suppose that $V(\mathfrak{n})$ contains at least two primes. Then there exists a unique regular A-scheme $\overline{\mathfrak{M}}_{\mathfrak{n}}$, proper over* $\operatorname{Spec} A$, *which contains the A-scheme $\mathfrak{M}_{\mathfrak{n}}$ as an open dense subscheme such that:*

(i) $\overline{\mathfrak{M}}_{\mathfrak{n}} \setminus \mathfrak{M}_{\mathfrak{n}} \to \operatorname{Spec} A$ *is finite,*

(ii) The completion of the scheme $\overline{\mathfrak{M}}_{\mathfrak{n}}$ along $\overline{\mathfrak{M}}_{\mathfrak{n}} \setminus \mathfrak{M}_{\mathfrak{n}}$ is canonically isomorphic to $\widehat{\mathfrak{M}}_{\mathfrak{n}}$.

(iii) $\overline{\mathfrak{M}}_{\mathfrak{n}} \to \operatorname{Spec} A$ *is smooth over* $\operatorname{Spec} A(\mathfrak{n})$ *of relative dimension one.*

Furthermore for $\mathfrak{n}' \subset \mathfrak{n}$, the covering map $\mathfrak{M}_{\mathfrak{n}'} \to \mathfrak{M}_{\mathfrak{n}}$ extends to a finite flat map $\overline{\mathfrak{M}}_{\mathfrak{n}'} \to \overline{\mathfrak{M}}_{\mathfrak{n}}$ with an action of $\mathcal{K}(\mathfrak{n})/\mathcal{K}(\mathfrak{n}')$, which is ramified possibly over $V(\mathfrak{n}')$ and ∞.

Let $\mathcal{K}$ be any admissible open subgroup of $\mathrm{GL}_2(\hat{A})$ *with minimal conductor $\mathfrak{n}$. Then, there exists a canonical smooth compactification $\overline{\mathfrak{M}}_{\mathcal{K}}$ of $\mathfrak{M}_{\mathcal{K}}$ over* $\operatorname{Spec} A(\mathfrak{n})$. *It is obtained as the quotient of $\overline{\mathfrak{M}}_{\mathfrak{n}}$ by $\mathcal{K}/\mathcal{K}(\mathfrak{n})$ and has properties analogous to (i) and (ii) over* $\operatorname{Spec} A(\mathfrak{n})$.

The A-morphism $\overline{\mathfrak{M}}_{\mathcal{K}} \to \operatorname{Spec} A(\mathfrak{n})$ is denoted by $\bar{g}_{\mathcal{K}}$, and similarly $\bar{g}_{\mathfrak{n}} : \overline{\mathfrak{M}}_{\mathfrak{n}} \to \operatorname{Spec} A(\mathfrak{n})$. The open immersion $\mathfrak{M}_{\mathcal{K}} \to \overline{\mathfrak{M}}_{\mathcal{K}}$ is denoted by $j_{\mathcal{K}}$.

PROOF: All the statements concerning $\overline{\mathfrak{M}}_{\mathfrak{n}}$ stem from [10], Prop. 9.3. So it remains to prove the assertions on $\overline{\mathfrak{M}}_{\mathcal{K}}$. Let $G := \mathcal{K}/\mathcal{K}(\mathfrak{n})$. By Theorem 1.15, we know that $\mathfrak{M}_{\mathcal{K}} \cong \mathfrak{M}_{\mathfrak{n}}/G$ is smooth over $\operatorname{Spec} A(\mathfrak{n})$. As the gluing is easily seen to be compatible with the action of $\mathcal{K}/\mathcal{K}(\mathfrak{n})$, and as $\widehat{\mathfrak{M}}_{\mathfrak{n}}/G \cong \widehat{\mathfrak{M}}_{\mathcal{K}}$, it remains to show that $\widehat{\mathfrak{M}}_{\mathcal{K}}$ is formally smooth over $\operatorname{Spec} A(\mathfrak{n})$. For this, we use its explicit description given in Lemma 2.11.

Thus, we need to consider the action of $s^{-1}\mathcal{K}s \cap B(\mathbb{A}^f)$ on $\mathfrak{N}$, or equivalently, making use of $\mathfrak{n}$, the action of $(s^{-1}\mathcal{K}s \cap B(\mathbb{A}^f))/V_{\mathfrak{n}}$ on $\mathfrak{N}_{\mathfrak{n},\mathfrak{n}}$. By Proposition 2.12, we may choose a set of double coset representatives s such that each s normalizes

$\mathrm{GL}_2(\hat{A})$. Thus it suffices to show that for any admissible $\mathcal{K}$, the action of $(\mathcal{K} \cap B(\mathbb{A}^f))/V_{\mathfrak{n}} = V_{\mathcal{K}}/V_{\mathfrak{n}}$ on $\mathfrak{N}_{\mathfrak{n},\mathfrak{n}}$ is free over $\operatorname{Spec} A(\mathfrak{n})$.

Define $\mathcal{K}_1 \subset \hat{A}^*$ to be the quotient $V_{\mathcal{K}}(1 \ltimes \mathbb{A}^f)/(1 \ltimes \mathbb{A}^f)$. Then $V_{\mathcal{K}} \subset \mathcal{K}_1 \ltimes \hat{A}$, and it will suffice to show that the action of $(\mathcal{K}_1 \ltimes \hat{A})/V_{\mathfrak{n}}$ is free. By Proposition 2.6 it is enough to show that $\mathcal{K}_1$ is admissible.

Assume that $\mathcal{K}_1$ is not admissible. As $\mathcal{K}_1$ is abelian, there must exist an element $a \in (k^* \setminus \{1\}) \cap \mathcal{K}_1$. Thus $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ will be in $\mathcal{K}(1 \ltimes \hat{A})$, i.e. $\mathcal{K}$ will contain an element of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \in \hat{A}$. If we conjugate this element with $t := \left(\begin{smallmatrix} 1 & \frac{b}{1-a} \\ 0 & 1 \end{smallmatrix}\right)$, we obtain

$$I \neq \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in t\mathcal{K}t^{-1} \cap \mathrm{GL}_2(k)$$

contradicting the admissibility of $\mathcal{K}$. ∎

3 Drinfeld's upper half plane

For the explicit description of the analytic moduli space for Drinfeld modules of rank 2 with some level structure to be given in the following section, we will review some necessary background in this section on Drinfeld's upper half plane Ω and its quotient by arithmetic subgroups. We first recall the description of the $\mathbb{C}_\infty$-valued points of Ω as a subset of $\mathbb{P}^1(\mathbb{C}_\infty)$ together with its reduction map to the Bruhat-Tits tree $\mathcal{T}$. Then we describe its structure as a rigid analytic space over K_∞. Finally we look at quotients $\Gamma\backslash\Omega$ and their canonical compactification for arithmetic subgroups Γ of $\mathrm{GL}_2(K)$. For an introduction to rigid analytic spaces, we refer to [5] and [12].

Two main objectives are, first, to give a self-contained exposition of the affinoid cover of $\Gamma\backslash\Omega$ that arises from a natural cover of $\Gamma\backslash\mathcal{T}$, and, second, to provide a useful affinoid cover for the canonical compactification of $\Gamma\backslash\Omega$.

A word on notation. We will use gothic letters for three different purposes. $\mathfrak{M}^?_?$ and $\mathfrak{N}^?_?$ were (and will be) used for algebraic moduli space and moduli spaces in general. Letters $\mathfrak{U}$, $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ will be used for rigid spaces. Finally, we also gothic letter do denote Čech covers. We hope that no confusion will arise.

3.1 The reduction map for $\Omega(\mathbb{C}_\infty)$

As a set of points, we define $\Omega(\mathbb{C}_\infty) := \mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus \mathbb{P}^1(K_\infty)$. This space has a similar meaning for the uniformization of certain rigid curves, as the complex upper half plane for compact Riemann surfaces of genus $g > 1$.

Following the treatment in [8], which is based on [10], we will now describe the reduction map ρ of $\Omega(\mathbb{C}_\infty)$. This will be useful when describing the rigid structure of Ω and its quotients. To describe ρ we will introduce the Bruhat-Tits tree $\mathcal{T}$ associated to rank 2 lattices of A_∞ and the set of norms on the vector space $V_\infty := K_\infty^2$ up to dilatations. The set $\{f_0, f_1\}$ will denote the standard basis of V_∞.

Definition 3.1 *An A_∞-*lattice *M of V_∞ is a free A_∞ submodule of V_∞ of rank 2.*

Two A_∞-lattices M, M' of V_∞ are equivalent*, if there exists some $a \in K_\infty^*$ such that $aM = M'$. An equivalence class is denoted by $[M]$.*

Definition 3.2 (The Bruhat-Tits tree $\mathcal{T}$) *The set $\mathcal{T}_0$ of vertices of $\mathcal{T}$ is the set of equivalence classes $[M]$ of A_∞-lattices of V_∞.*

The set $\mathcal{T}_1$ of edges of $\mathcal{T}$ is the set of pairs $\{v, v'\}$ of vertices which satisfy the following property: There exist lattices $M \subset M'$ of index q_∞ such that $v = [M]$ and $v' = [M']$.

The elements of $\mathcal{T}_0 \cup \mathcal{T}_1$ are called the simplices *of $\mathcal{T}$.*

The above definition of $\mathcal{T}_1$ does not involve any kind of orientation on $\mathcal{T}$, because if $M \subset M'$ has index q_∞, then so does $M' \subset \pi_\infty^{-1} M$ and $[M] = [\pi_\infty^{-1} M]$.

It is simple to see that $\mathcal{T}$ is a connected tree and that there are $q_\infty + 1$ edges originating from every vertex of $\mathcal{T}$, cf. [48], II.1.1.

By a path, we mean a sequence of distinct vertices $v_0, \ldots, v_n$ such that for all i, $\{v_i, v_{i+1}\}$ is an edge.

Definition 3.3 *By $\mathcal{T}^o$, we define the oriented graph associated to $\mathcal{T}$: It has the same vertices as $\mathcal{T}$. For any edge $\bar{e}$ connecting vertices v, v', the graph $\mathcal{T}^o$ has an oriented edge $\overrightarrow{vv'}$ from v to v' and an oriented edge $\overrightarrow{v'v}$ in the opposite direction.*

We say that two oriented edges e, e' of $\mathcal{T}^o$ have the same orientation, *if there exists a path $v_0, \ldots, v_n$ in $\mathcal{T}$ such that either $e = \overrightarrow{v_0 v_1}$ and $e' = \overrightarrow{v_{n-1} v_n}$ or $e' = \overrightarrow{v_0 v_1}$ and $e = \overrightarrow{v_{n-1} v_n}$.*

For an oriented edge e, define $t(e) \in \mathcal{T}_0$ as the target of e. If e is an oriented edge, then by $-e$, we define the oriented edge with the opposite orientation and the same underlying (non-oriented) edge.

An orientation of $\mathcal{T}$ *is a choice of an oriented edge of $\mathcal{T}^o$ for each edge of $\mathcal{T}$, i.e., a splitting of the canonical surjection $\mathcal{T}_1^o \twoheadrightarrow \mathcal{T}_1$.*

To distinguish notationally between edges and oriented edges, the former are overlined, and the latter ones are not.

The *geometric realization* $|\mathcal{T}|$ of the graph $\mathcal{T}$ is defined as follows: To each edge $\bar{e} = \{v, v'\}$ we assign an interval

$$\mathrm{i}_{\bar{e}} := \{(\alpha_v, \alpha_{v'}) : \alpha_v + \alpha_{v'} = 1, 0 \leq \alpha_v, \alpha_{v'}\} \cong [0, 1].$$

One then defines $|\mathcal{T}| := \coprod_{\bar{e} \in \mathcal{T}_1} \mathrm{i}_{\bar{e}} / \sim$, where $\sim$ is the equivalence relation that identifies for each $v \in \mathcal{T}_0$ the points $(\alpha_v = 1, \alpha_{v'} = 0)$ such that $\{v, v'\} \in \mathcal{T}_1$.

Definition 3.4 *A point on $\mathrm{i}_{\bar{e}}$ is called* rational *if its coordinates α_v and $\alpha_{v'}$ are rational. A point $t \in |\mathcal{T}|$ is called rational, if t is a rational point on some interval $\mathrm{i}_{\bar{e}}$.*

As $\mathcal{T}$ is a tree, it is quite obvious how to define a canonical distance function $d(x, x')$ for $x, x' \in |\mathcal{T}|$ such that the distance between neighboring vertices is one. This makes $|\mathcal{T}|$ into a metric space.

We now come to the definition of norms on V_∞ modulo dilatations. This provides a natural link between $\Omega(\mathbb{C}_\infty)$ and $|\mathcal{T}|$.

Definition 3.5 *A* norm *on V_∞ is a map $\nu : V_\infty \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in V_\infty$ and $\alpha \in K_\infty$ the following hold: a) $\nu(x + y) \leq \max\{\nu(x), \nu(y)\}$, b) $\nu(\alpha x) = |\alpha|_\infty \nu(x)$, and c) $\nu(x) = 0$ if and only if $x = 0$.*

A norm ν' is called a dilatation *of a norm ν, if there exists $r \in \mathbb{R}_{>0}$ such that $\nu' = r\nu$. The dilatation class of ν is denoted by $[\nu]$, the set of all such classes by $N(V_\infty)$.*

Given dilatation classes $[\nu], [\nu']$, we define their distance as

$$\delta([\nu], [\nu']) := \log_{q_\infty} \sup_{x \neq 0} \nu(x)/\nu'(x) + \log_{q_\infty} \sup_{x \neq 0} \nu'(x)/\nu(x).$$

Note that this definition is independent of the chosen representatives and one can check that $N(V_\infty)$ together with δ is a metric topological space.

Next we describe natural maps which complete the diagram

$$\Omega(\mathbb{C}_\infty) \xrightarrow{\tilde{\rho}} N(V_\infty) \xleftarrow{\theta} |\mathcal{T}|. \tag{7}$$

Given $(z_0 : z_1) \in \Omega(\mathbb{C}_\infty)$, we define the norm $\nu_{(z_0, z_1)}$ by

$$\nu_{(z_0, z_1)}(\alpha_0 f_0 + \alpha_1 f_1) := |\alpha_0 z_0 + \alpha_1 z_1|_{\mathbb{C}_\infty}.$$

Based on the fact that $(z_0 : z_1) \notin \mathbb{P}^1(K_\infty)$, it is simple to see that this defines a norm. If one chooses a different representative of the point $\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) \in \mathbb{P}^1(\mathbb{C}_\infty)$, one obtains a dilatation of $\nu_{(z_0, z_1)}$. Hence we obtain a well-defined map $\tilde{\rho}$ as described above. Because $|\,.\,|_{\mathbb{C}_\infty}$ takes values in $q_\infty^{\mathbb{Q}}$, the dilatation classes in the image of $\tilde{\rho}$ have a representative which takes its values in $q_\infty^{\mathbb{Q}}$.

We now describe θ: Given a lattice M, we define

$$\nu_M(x) := \inf\{|\alpha|_\infty : \alpha \in K_\infty, x \in \alpha M\}.$$

It is simple to verify that this defines a norm. Given lattices $M \subset M'$ of index q_∞ and $\alpha_M, \alpha_{M'} \in \mathbb{R}_{\geq 0}$ with $\alpha_M + \alpha_{M'} = 1$, we define the norm

$$\nu_{M,\alpha_M,M',\alpha_{M'}}(x) := \sup\{\nu_{M'}(x), q_\infty^{-\alpha_{M'}}\nu_M(x)\}.$$

These assignments induce maps from $\mathcal{T}_0$ to $N(V_\infty)$ and similarly $\theta : |\mathcal{T}| \longrightarrow N(V_\infty)$. The following can be found in [8], Ch. 3:

Theorem 3.6 (Drinfeld) *The map θ is an isomorphism of metric spaces. If one defines*

$$\rho := \tilde{\rho}\theta^{-1} : \Omega(\mathbb{C}_\infty) \to |\mathcal{T}|,$$

then this map surjects onto the rational points of $|\mathcal{T}|$.

The map ρ is called the *reduction map* of $\Omega(\mathbb{C}_\infty)$.

3.2 The action of $\mathrm{GL}_2(K_\infty)$

Our next goal is to describe an action of $\mathrm{GL}_2(K_\infty)$ on the objects in diagram (7) such that the morphisms θ and $\tilde{\rho}$ are $\mathrm{GL}_2(K_\infty)$-equivariant.

We fix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$. Its action on $\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) \in \Omega$ is defined by

$$\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right) \mapsto \gamma\left(\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right).$$

Elements of V_∞ are considered as row vectors x on which γ acts via $x \mapsto \gamma \circ x := x\gamma^{-1}$. For a norm ν we define

$$(\gamma\nu)(x) := \nu(x\gamma),$$

which is in line with the idea that a norm is a 'morphism' from a space on which we act by $\mathrm{GL}_2(K_\infty)$ to a space with a trivial action. Finally, we define an action of γ on A_∞-lattices of V_∞ by sending the lattice M to $\gamma \circ M := \{\gamma \circ m : m \in M\}$, which is again an A_∞-lattice of V_∞. It is simple to verify that this induces an action of γ on $\mathcal{T}$, $\mathcal{T}^o$ and $|\mathcal{T}|$.

Proposition 3.7 *The above defines left actions of the group $\mathrm{GL}_2(K_\infty)$ on $\Omega(\mathbb{C}_\infty)$, $N(V_\infty)$, $\mathcal{T}$, $\mathcal{T}^o$ and $|\mathcal{T}|$ which factor through $\mathrm{PGL}_2(K_\infty)$. With respect to these actions, the maps θ and $\tilde{\rho}$ are $\mathrm{GL}_2(K_\infty)$-equivariant.*

Definition 3.8 *The* standard vertex *is $v_0 := [A_\infty f_0 \oplus A_\infty f_1]$. By v_1 we denote the vertex $[\pi_\infty A_\infty f_0 \oplus A_\infty f_1]$. The* standard oriented edge *is $e_0 := \overrightarrow{v_0 v_1}$.*

The inverse image of v_0 under the reduction map ρ is the set

$$\{z \in \mathcal{O}_{\mathbb{C}_\infty} : z \pmod{\mathfrak{m}_{\mathbb{C}_\infty}} \notin k_\infty\}$$

which is simply the closed unit disk in $\mathbb{C}_\infty$ with q_∞ open discs of radius one around the points of k_∞ removed.

Proposition 3.9 *The action of $\mathrm{GL}_2(K_\infty)$ is transitive on vertices, as well as on oriented edges. The stabilizer of the action on v_0 is $\mathrm{GL}_2(A_\infty)K_\infty^*$. The stabilizer of e_0 is $K_\infty^*\Gamma_{e_0}$, where*

$$\Gamma_{e_0} := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A_\infty) : c \in \pi_\infty A_\infty\right\}.$$

3.3 Ω as a rigid space over K_∞

We will now use the reduction map to construct a rigid space Ω whose points over $\mathbb{C}_\infty$ are in bijection with $\Omega(\mathbb{C}_\infty)$. The set $\Omega(\mathbb{C}_\infty)$ is regarded as a subset of $\mathbb{C}_\infty$ via the map $(z_1 : z_2) \mapsto z_1/z_2$. We first define a Čech cover of $|\mathcal{T}|$.

Definition 3.10 *Define $W_{v_0} := \{p \in |\mathcal{T}| : d(p, v_0) \leq 1/3\}$ and*

$$W_{\bar{e}_0} := \{(v_0, \alpha, v_1, \beta) : \alpha \geq 1/3, \beta \geq 1/3, \alpha + \beta = 1\}.$$

Furthermore for a general vertex $v = \gamma v_0$ set $W_v := \gamma W_{e_0}$ and for an edge $\bar{e} = \gamma \bar{e}_0$ set $W_{\bar{e}} := \gamma W_{\bar{e}_0}$.

The set W_{v_0} is a star around v_0 where each of the $q_\infty + 1$ line segments originating from v_0 has length $1/3$, and W_{e_0} is a line segment of length $1/3$. The definitions of W_v and W_e are independent of the chosen γ. The set $\mathfrak{W} := \{W_t : t \in \mathcal{T}\}$ is a Čech cover of $|\mathcal{T}|$ by compact connected subsets of $|\mathcal{T}|$. The next result, cf. [8], shows how $\rho^{-1}(\mathfrak{W})$ gives rise to an affinoid cover of $\Omega(\mathbb{C}_\infty)$ which is already defined over K_∞.

Proposition 3.11 *For each connected compact subset of $\mathcal{T}$ with rational end-points its preimage under ρ is the set of the $\mathbb{C}_\infty$-valued points of an affinoid subset of $\mathbb{P}^1_{K_\infty}$.*

For $t \in \mathcal{T}$, we define $\mathfrak{U}_t(\mathbb{C}_\infty) := \rho^{-1}(W_t)$. For $t = v_0$ and $t = e_0$, one has the following explicit description of $\mathfrak{U}_t$:

$$\begin{aligned}
\mathfrak{U}_{v_0}(\mathbb{C}_\infty) &= \left\{ z \in \mathbb{C}_\infty : |z - \beta|_\infty \geq q_\infty^{-1/3} \text{ for all } \beta \in k_\infty \text{ and } |z|_\infty \leq q_\infty^{1/3} \right\}, \\
\mathfrak{U}_{e_0}(\mathbb{C}_\infty) &= \left\{ z \in \mathbb{C}_\infty : q_\infty^{1/3} \leq |z|_\infty \leq q_\infty^{2/3} \right\}.
\end{aligned}$$

Via the action of $\mathrm{GL}_2(K_\infty)$, one can obtain the remaining $\mathfrak{U}_t$ from $\mathfrak{U}_{v_0}$ and $\mathfrak{U}_{e_0}$, and it follows from these explicit descriptions that the sets $\mathfrak{U}_t$ are $\mathbb{C}_\infty$-valued points of affinoids defined over K_∞. In [12], V.1, the following result due to Drinfeld is explained.

Proposition 3.12 *The cover $\mathfrak{U} := \{\mathfrak{U}_t : t \in \mathcal{T}\}$ is an admissible affinoid cover of a rigid analytic space Ω whose set of $\mathbb{C}_\infty$-valued points is $\mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus \mathbb{P}^1(K_\infty)$.*

For any affinoid space X, the ring of holomorphic sections on X is denoted by $\mathcal{O}_X$. In particular, it is easy to see that all rational functions with poles in $\mathbb{P}(K_\infty)$ lie in $\mathcal{O}_\Omega$.

3.4 Arithmetic subgroups of $\mathrm{GL}_2(K)$

For a non-zero ideal $\mathfrak{n}$ of A, let $\Gamma(\mathfrak{n})$ denote the standard congruence subgroup of $\mathrm{GL}_2(A)$ of level $\mathfrak{n}$, i.e., the subgroup of all matrices which are the identity modulo $\mathfrak{n}$. Following [15], we make the following definition.

Definition 3.13 *A subgroup Γ of $\mathrm{GL}_2(K)$ is called an* arithmetic subgroup *if there exists an ideal $\mathfrak{n}$ of A such that Γ contains $\Gamma(\mathfrak{n})$ and such that this inclusion is of finite index.*

The following gives an alternative characterization of such subgroups, which is used in [15], V.2, as their definition:

Proposition 3.14 *A subgroup Γ of $\mathrm{GL}_2(K)$ is an arithmetic subgroup, if and only if there exists a projective A-submodule Λ of K^2 of rank 2 and an ideal $\mathfrak{n}$ of A such that*

$$\mathrm{Aut}_A(\Lambda) \supset \Gamma \supset \mathrm{Aut}_A(\Lambda, \mathfrak{n}) := \{\gamma \in \mathrm{Aut}_A(\Lambda) : \gamma \equiv \mathrm{id} \pmod{\mathfrak{n}\Lambda}\},$$

where $\mathrm{Aut}_A(\Lambda)$ is the set of A-linear automorphisms of Λ.

Proof: Let first Γ be an arithmetic subgroup of $\mathrm{GL}_2(K)$ which contains the group $\Gamma(\mathfrak{n}')$ as a subgroup of finite index for some non-zero ideal $\mathfrak{n}'$ of A. We will construct Λ and $\mathfrak{n}$ with the desired properties: We define $\Lambda := \bigcap_{\gamma\in\Gamma} \gamma A^2$. Because $A^2 = \gamma A^2$ for all $\gamma \in \Gamma(\mathfrak{n}')$, this is really a finite intersection of torsion free A-modules of rank 2 inside K^2. It follows easily that Λ must be a projective A-module of rank 2 for which $\Gamma \subset \mathrm{Aut}_A(\Lambda)$.

As Λ is of rank 2, we can choose a non-zero ideal $\mathfrak{n} \subset \mathfrak{n}'$ of A such that $\mathfrak{n}A^2 \subset \Lambda$. Hence

$$\mathfrak{n}\Lambda \subset \mathfrak{n}A^2 \subset \Lambda \subset A^2$$

and it follows that $\mathrm{Aut}_A(\Lambda, \mathfrak{n}^2) \subset \Gamma(\mathfrak{n}') \subset \Gamma$.

For the other direction, suppose we are given $\mathfrak{n}$ and Λ. We may clearly assume that $\Lambda \subset A^2$. Choose $\mathfrak{n}' \subset \mathfrak{n}$ such that $\mathfrak{n}'A^2 \subset \Lambda$. Then $\Gamma \supset \mathrm{Aut}_A(\Lambda, \mathfrak{n}) \supset \Gamma(\mathfrak{n}^2)$, and it follows readily that Γ is an arithmetic subgroup of $\mathrm{GL}_2(K)$. ∎

As the set of A-lattices $\Lambda \subset K^2$ of rank 2 is invariant under conjugation by elements in $\mathrm{GL}_2(K)$, the following result is clear.

Corollary 3.15 *If Γ is an arithmetic subgroup of $\mathrm{GL}_2(K)$, then so is $\gamma\Gamma\gamma^{-1}$ for any $\gamma \in \mathrm{GL}_2(K)$.*

Proposition 3.16 *Suppose that Γ is an arithmetic subgroup of $\mathrm{GL}_2(K)$. Then $\det(\Gamma) \subset k^* \subset K^*$ and furthermore, the action of Γ on $\mathcal{T}$ is orientation preserving, i.e., for any $\gamma \in \Gamma$ and any vertex v of $\mathcal{T}$, the distance between v and $\gamma(v)$ is an even integer.*

Proof: Since the index of $\Gamma \supset \Gamma(\mathfrak{n})$ is finite, so is the index of the subgroups $\det(\Gamma) \supset \det(\Gamma(\mathfrak{n})) = \{1\}$ of K^*. But the only finite subgroups of K^* are subgroups of k^*. This shows the first assertion. The second follows from [48], p. 104 f., which says that any Γ which satisfies $\det \Gamma \subset k^*$ is orientation preserving. (The reason behind this is that $\mathrm{v}_\infty(\det \gamma) \equiv 0 \pmod 2$ provided that $\det \gamma \in k^*$.) ∎

3.5 Quotients of $\mathcal{T}$

Our goal is to describe $\Gamma\backslash\Omega$ and its $\mathbb{C}_\infty$-valued points. As the reduction map is crucial for the understanding of this quotient, we first recall some facts about $\Gamma\backslash|\mathcal{T}|$.

For $t \in \mathcal{T}$, we define its stabilizer in Γ as $\Gamma_t := \{\gamma \in \Gamma : \gamma(t) = t\}$.

Lemma 3.17 *For $t \in \mathcal{T}$, the group Γ_t is finite.*

Proof: For $\Gamma \subset \mathrm{GL}_2(A)$, this can be found in [48], II.1. For general Γ, one uses the fact that Γ and $\mathrm{GL}_2(A)$ are commensurable. ∎

By Proposition 3.16, the stabilizer of any edge t acts trivially on its geometric realization in $|\mathcal{T}|$. On the star-shaped regions W_t, $t \in \mathcal{T}_0$, the action of Γ_t fixes the vertex t but may permute the edges emanating from t.

It is a simple exercise to construct the quotient graph $\Gamma\backslash\mathcal{T}$ and to show that its geometric realization agrees with the quotient $\Gamma\backslash|\mathcal{T}|$. Note that $\Gamma\backslash\mathcal{T}$ may have multiple edges between two vertices. (However no edge may start and end at the same vertex.) Before we give a useful description of $\Gamma\backslash\mathcal{T}$, we review the notion of an end of $\mathcal{T}$.

Definition 3.18 *A* half line *of $\mathcal{T}$ is a sequence $\underline{s} := \{[M_i]\}_{i\in\mathbb{N}_0}$ of vertices of $\mathcal{T}$ such that for each $i > 0$ the vertices $[M_{i-1}]$ and $[M_{i+1}]$ are adjacent to $[M_i]$ and distinct.*

Two half lines $\underline{s}$ and $\underline{s}'$ are said to be equivalent *if there exists $j, j' \geq 0$ such that $[M_{i+j}] = [M'_{i+j'}]$ for all $i \in \mathbb{N}_0$. An* end *is an equivalence class $[\underline{s}]$ of a half line $\underline{s}$.*

Since $\mathcal{T}$ is a tree, any end has a unique representative $\underline{s}$ which starts at v_0. Using the elementary divisor theorem, one can see that there exists a vector $\underline{l}$ in $M_0 := A_\infty f_0 \oplus A_\infty f_1$ which is unique up to multiplication by an element in A_∞^* such that $\underline{s}$ is given by the sequence of lattices $M_i := \underline{l}A_\infty + \pi_\infty^i M_0$. For the following proposition see [48], II.1.1.

Because we had defined a left action of $\gamma \in \mathrm{GL}_2(K_\infty)$ on the row vector $v \in V_\infty \cong K_\infty{}^2$ by $\gamma \circ v = v\gamma^{-1}$, we define a left action of $\gamma \in \mathrm{GL}_2(K_\infty)$ on $(x_1 : x_2) \in \mathbb{P}^1(K_\infty)$ by viewing $(x_1 : x_2)$ as a row vector and acting on with γ^{-1} from the right.

Proposition 3.19 *The map which sends the end $[\underline{s}]$ to the line in $\mathbb{P}^1(K_\infty)$ generated by $\underline{l}$ induces a $\mathrm{GL}_2(K_\infty)$-equivariant bijection between the ends of $\mathcal{T}$ and $\mathbb{P}^1(K_\infty)$.*

Definition 3.20 *An end is called* rational *if it corresponds to an element in $\mathbb{P}^1(K)$ under the above bijection.*

The elements of $\Gamma\backslash\mathbb{P}^1(K)$, i.e., the equivalence classes of rational ends modulo Γ, are called the cusps *of $\Gamma\backslash\mathcal{T}$.*

The following theorem describes the structure of $\Gamma\backslash\mathcal{T}$:

Theorem 3.21 *Let Γ be an arithmetic subgroup of $\mathrm{GL}_2(A)$. Then $\Gamma\backslash\mathcal{T}$ is the union of a finite connected subgraph $\mathbb{Y}$ and subgraphs Δ_x for each cusp $x \in \Gamma\backslash\mathbb{P}^1(K)$ such that the following hold:*

(i) Each Δ_x is a graph with vertices $\{v_{x,i} : i \geq 0\}$ and edges $\{\bar{e}_{x,i} : i > 0\}$ where the edge $\bar{e}_{x,i}$ connects the two vertices $v_{x,i-1}$ and $v_{x,i}$. Furthermore, Δ_x is represented by a half line $\underline{s}$ of $\mathcal{T}$ whose corresponding end is in the equivalence class of the cusp x.

(ii) For cusps $x \neq x'$, the graphs Δ_x and $\Delta_{x'}$ are disjoint.

(iii) The subgraphs $\mathbb{Y}$ and Δ_x have precisely one vertex in common, namely $v_{x,0}$.

Note that for a cusp x, the geometric realization of Δ_x is isomorphic to $\mathbb{R}_{\geq 0}$.

Proof: Let Λ be an A-submodule of K^2 as in Proposition 3.14, so that $\Gamma \subset \mathrm{Aut}_A(\Lambda)$. If $\Lambda \cong A^2$, then the theorem is given in [48], II.2.3, cf. in particular Thm. 9.

If $\Gamma = \mathrm{Aut}_A(\Lambda)$, but Λ is arbitrary, the asserted result follows by modifying [48], pp.130–146, as follows: In II.2.1, on defines E_L as the vector bundle on C

which agrees with Λ, when restricted to Spec(A), and has L as the fiber above ∞ (the place P from loc. cit. corresponds to ∞ in our notation). Let F_Γ be the unique (up to isomorphism) line bundle on C which agrees with $\Lambda \wedge \Lambda$, when restricted to $\operatorname{Spec} A$, and satisfies $0 \geq \deg F_\Gamma > -d_\infty$. Then Prop. 5 in II.2.1 asserts a bijection between the vertices of $\Gamma\backslash\mathcal{T}$ and the vector bundles on C whose determinant is isomorphic to either F_Γ or $F_\Gamma \otimes I_\infty$.

II.2.2 remains unchanged. Finally, II.2.3 needs the following changes. Define $F'_{c,n} := I_P^{\otimes n} \otimes F_c^{-1} \otimes F_\Gamma$. II.Prop 9 remains true if one imposes the condition $(n-1)d_\infty > 2g - 2 - 2f_c + \deg F_\Gamma$. On page 143 one needs to define $m := \max\{2g - 2 + d_\infty, 3d_\infty - 2 - \deg\ F_\Gamma\}$ and n_c as the largest integer such that $2f_c + n_c d_\infty \leq m + \deg F_\Gamma$. Then II.Thm.9 remains valid and the proof of the above theorem is completed.

The case where Γ is a proper subgroup of $\mathrm{Aut}_A(\Gamma)$, follows from the remarks in [48], p. 172. ∎

For later use, we now recall the notion of stable and unstable simplices. It will be important when we review Teitelbaum's description of modular forms and furthermore for defining an affinoid cover of the compactification of $\Gamma\backslash\Omega$. All results which we do not prove are from [48], p. 176ff.

Let Γ denote an arithmetic subgroup of $\mathrm{GL}_2(K)$ which is p'-torsion free.

Definition 3.22 *A simplex $t \in \mathcal{T}$ is called* stable (with respect to Γ) *if Γ_t is trivial. Otherwise it is called* unstable.

If an edge is unstable, then the vertices on both ends are unstable. Therefore the unstable simplices form a subgraph $\mathcal{T}_\infty$ of $\mathcal{T}$ which is a countable disjoint union of trees. By Lemma 3.23, there exists a map $b\colon \mathcal{T}_\infty \to \mathbb{P}^1(K)$ which sends an unstable simplex t to the unique rational end $[\underline{s}]$ such that $\Gamma_t \subset \Gamma_{\underline{s}}$. It is constant on connected components and thus b gives a labeling of the subtrees of $\mathcal{T}_\infty$.

The group Γ acts on $\mathcal{T}_\infty$ and, by Theorem 3.21, the quotient $\Gamma\backslash\mathcal{T}_\infty$ is a finite disjoint union of $\mathrm{card}(\Gamma\backslash\mathbb{P}^1(K))$ graphs each of which contains precisely one cusp of $\Gamma\backslash\mathcal{T}$. One can also show purely combinatorially that all connected components of the geometric realization of $\Gamma\backslash\mathcal{T}_\infty$ are contractible.

Based on the finiteness of Γ_t, one can show the following lemma, cf. [48], II.2.9:

Lemma 3.23 *Suppose Γ is p'-torsion free. If Γ_t is non-trivial, then there exists a unique rational end $[\underline{s}]$ whose stabilizer $\Gamma_{\underline{s}} \subset \Gamma$ contains Γ_t.*

3.6 Quotients of Ω

To obtain $\Gamma\backslash\Omega$, one first constructs the affinoids $\Gamma_t\backslash\mathfrak{U}_t$. Their existence as rigid spaces is guaranteed by the following lemma due to Drinfeld, cf. [5], 6.3.3, Prop. 3. Below we will give an explicit construction of these quotients.

Lemma 3.24 *Let B be an affinoid algebra with associated space* $\mathrm{Spm}(B)$ *on which a finite group G acts. Then the quotient space* $\mathrm{Spm}(B)/G$ *exists and is given by* $\mathrm{Spm}(B^G)$.

The stabilizers of a non-trivial intersection $W_{tt'} := W_t \cap W_{t'}$ is given by $\Gamma_{tt'} := \Gamma_t \cap \Gamma_{t'}$. If $W_{tt'} \neq \varnothing$, one of t, t' is an edge e, the other an adjacent vertex v and one has $\Gamma_{tt'} = \Gamma_e \subset \Gamma_v$. One may therefore glue the affinoids $\Gamma_t\backslash\mathfrak{U}_t$ along their intersections $\Gamma_{tt'}\backslash\mathfrak{U}_{tt'}$, where $\mathfrak{U}_{tt'} = \mathfrak{U}_t \cap \mathfrak{U}_{t'}$. Furthermore, one may verify that the reduction map induces reduction maps $\Gamma_t\backslash\mathfrak{U}_t \to \Gamma_t\backslash W_t$ which can be glued.

Let D^* denote the punctured affinoid unit disc defined over K_∞.

Theorem 3.25 *For any arithmetic subgroup Γ, there exists a rigid quotient space $\Gamma\backslash\Omega$ of Ω by Γ with the usual universal property of a quotient and an induced reduction map $\Gamma\backslash\Omega(\mathbb{C}_\infty) \to |\Gamma\backslash\mathcal{T}|$, denoted by ρ_Γ. The map ρ_Γ satisfies the following conditions:*

(i) Let $t \in \mathcal{T}_1$ be an edge with endpoints v, v' and E_t a closed line segment of $\mathfrak{i}_t$ with rational endpoints distinct from v, v'. Then the preimage of $\Gamma_t\backslash E_t$ is a closed annulus.

(ii) Let t be in $\mathcal{T}_0$ and for a rational number $\alpha \in (0,1)$ define the star $S_t(\alpha)$ with center t as $\{p \in |\mathcal{T}| : d(p,t) \leq \alpha\}$. Then the preimage of $\Gamma_t\backslash S_t(\alpha)$ is isomorphic to the closed unit disc with r_t disjoint open subdiscs removed, where $r_t + 1$ is the number of edges emanating from the vertex $\Gamma_t\backslash t$.

(iii) For any cusp x and any $i \geq 0$ let $C_{x,m}$ denote the geometric realization of the subgraph of Δ_x with vertices $v_{x,i}$, $i \geq m$ and edges $\bar{e}_{x,i}$, $i > m$. Then there exists an m_0 such that for all $m \geq m_0$ the preimage of $C_{x,m}$ under ρ_Γ is isomorphic to $D^*(\mathbb{C}_\infty)$.

Furthermore, $\mathfrak{U}_\Gamma := \{\Gamma_t\backslash\mathfrak{U}_t : t \in \mathcal{T}\}$ is an admissible cover of $\Gamma\backslash\Omega$ defined over K_∞.

Proof: As we lack a satisfactory reference, we sketch a proof of the above result, which seems to be well-known, cf. [42], Thm.2.1. We restrict ourselves to the case where Γ is p'-torsion free, which is the only case which will matter later on. For the general case observe that any arithmetic subgroup Γ' contains a p'-torsion free arithmetic subgroup Γ of finite index. Thus to derive the above theorem for Γ', one may describe $\Gamma'\backslash\Omega$ as the quotient of $\Gamma\backslash\Omega$ by the finite group Γ'/Γ. Details are left to the interested reader.

We will only sketch parts (ii) and (iii), as (i) is similar to, but simpler than case (ii). We will explain the admissibility of $\mathfrak{U}_\Gamma$ when discussing the compactification of $\Gamma\backslash\Omega$, cf. Remark 3.30. Let us fix t such that Γ_t is non-trivial. Using Lemma 3.23, there exists a unique rational end $[\underline{s}]$ of $\mathbb{P}^1(K)$, such that $\Gamma_t \subset \Gamma_{\underline{s}}$. Furthermore, pick $\gamma \in \mathrm{GL}_2(K)$ such that $\gamma[\underline{s}] = (0:1) \in \mathbb{P}^1(K)$. By replacing Γ with $\gamma\Gamma\gamma^{-1}$, which by Corollary 3.15 is again an arithmetic subgroup, we may assume $[\underline{s}] = (0:1)$.

Definition 3.26 *We call an additive subgroup I of $(K,+)$ a* fractional almost-ideal, *if I contains a fractional ideal of A of finite index.*

As Γ is p'-torsion free, there exists a fractional almost-ideal $I_{\underline{s}}$ of A such that $\Gamma_{\underline{s}} = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} : b \in I_{\underline{s}}\right\}$. Define I_t to be the finite additive subgroup of $I_{\underline{s}}$ such that $\Gamma_t = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} : b \in I_t\right\}$.

Any vertex can be represented by a lattice whose basis is given by the row vectors $(1,y)$ and $(0,\pi_\infty^n)$ where $n \in \mathbb{Z}$ is unique and $y \in K_\infty$ is unique modulo π_∞^n, so we assume t to be given by such a lattice. Then one has

$$\begin{aligned}\mathrm{Stab}_{\mathrm{GL}_2(K_\infty)}(t) &= \mathrm{Stab}_{\mathrm{GL}_2(K_\infty)}\left(\begin{pmatrix} 1 & y \\ 0 & \pi_\infty^n\end{pmatrix}^{-1} v_0\right)\\ &= \begin{pmatrix} 1 & -y\pi_\infty^{-n} \\ 0 & \pi_\infty^{-n}\end{pmatrix}\mathrm{GL}_2(A_\infty)K_\infty^*\begin{pmatrix} 1 & y \\ 0 & \pi_\infty^n\end{pmatrix}.\end{aligned}$$

Because Γ_t is of the form given above, an explicit calculation shows that

$$\Gamma_t = \Gamma_{\underline{s}} \cap \mathrm{Stab}_{\mathrm{GL}_2(K_\infty)}(t) = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} : b \in \pi_\infty^n A_\infty \cap I_{\underline{s}}\right\},$$

which does not depend on y. In particular, $I_t = \pi_\infty^n A_\infty \cap I_{\underline{s}}$ and typically $n \leq 0$.

We now turn to the proof of (ii), where we assume that Γ_t is non-trivial. The preimage of $S_t(\alpha)$ under ρ is given by

$$\begin{aligned} S_\alpha \quad := \quad & \big\{ z \in \mathbb{C}_\infty : |z - y - \beta \pi_\infty^n|_\infty \geq |\pi_\infty|_\infty^{n+\alpha} \text{ for } \beta \in k_\infty \text{ and} \\ & \qquad |z - y|_\infty \leq |\pi_\infty|_\infty^{n-\alpha} \big\} . \end{aligned}$$

We consider the additive map $f : z \mapsto \prod_{i \in I_t}(z+i) : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Clearly the fiber of each $w = f(z)$ is the orbit under Γ_t of an element z. We will determine $f(S_\alpha)$ and show that the map $S_\alpha \to f(S_\alpha)$ identifies $f(S_\alpha)$ with the quotient $\Gamma_t \backslash S_\alpha$. The explicit shape of $f(S_\alpha)$ will finish the proof of (ii).

Define $J_1 := \pi_\infty^{-n} I_t \cap \pi_\infty A_\infty$ and let $J_0 \subset k_\infty$ be an additive set of representatives of $\pi_\infty^{-n} I_t$ modulo J_1. Then $I_t = \pi_\infty^n (J_0 + J_1)$ and for $\beta \in k_\infty$ one obtains:

$$\begin{aligned} |w - f(y + \beta \pi_\infty^n)|_\infty &= |f(z - y - \beta \pi_\infty^n)|_\infty && (8) \\ &= \prod_{j_1 \in J_1} \prod_{j_0 \in J_0} |z - y - (\beta + j_0 + j_1) \pi_\infty^n|_\infty && (9) \\ &= \prod_{j_0 \in J_0} |z - y - (\beta + j_0) \pi_\infty^n|_\infty^{|J_1|} && (10) \\ &\geq |\pi_\infty|_\infty^{n|I_t| + \alpha |J_1|}, && (11) \end{aligned}$$

where we use in (10) that $|z - y - (\beta + j_1 + j_0)\pi_\infty^n|_\infty \geq |\pi_\infty|_\infty^{n+\alpha} > |j_1 \pi_\infty^n|_\infty$ and in (11) that at most one of the factors in (10) is of absolute value less than $|\pi_\infty^n|_\infty^{|J_1|}$.

Let T_α be the affinoid

$$T_\alpha := \{ w : |w - f(y + \beta \pi_\infty^n)|_\infty \geq |\pi_\infty|_\infty^{n|I_t| + \alpha|J_1|} \text{ for all } \beta \in k_\infty \text{ and}$$

$$|w - f(y)|_\infty \leq |\pi_\infty|_\infty^{(n-\alpha)|I_t|} \}.$$

To prove surjectivity of $f : S_\alpha \to T_\alpha$, note first that for any w we can find a solution $z \in \mathbb{C}_\infty$ such that $f(z) = w$. We leave it to the reader to show that for $z \in \mathbb{C}_\infty$ the element $f(z)$ cannot satisfy the inequalities that describe T_α unless $z \in S_\alpha$.

Note that the description of $f(S_\alpha)$ only involves $1 + |k_\infty|/|J_0|$ inequalities as for $\beta, \beta_2 \in k_\infty$ the inequalies

$$|z - f((y + \beta_i)\pi_\infty^n)|_\infty \geq |\pi_\infty|_\infty^{n|I_t| + \alpha|J_1|}, i = 1, 2$$

describe the same set precisely when $\beta_1 - \beta_2 \in J_0$. At the same time one may verify that the number of edges in $|\Gamma \backslash \mathcal{T}|$ originating from t is also given by $1 + |k_\infty|/|J_0|$.

So far, we only checked that the map f is finite-to-one between the sets of points S_α and T_α. Because S_α is I_t-invariant, we have $S_\alpha = f^{-1}(T_\alpha)$. To conclude the proof of (ii), we use the following lemma, whose proof is obvious and left to the reader:

Lemma 3.27 *The morphism of schemes $\mathbb{A}^1 \to \mathbb{A}^1$ corresponding the ring map $k[w] \to k[z] : w \mapsto \prod_{i \in I_t}(z+i)$ is a finite étale cover with covering group I_t.*

The induced morphism of rigid spaces $f \colon \mathbb{A}^{1,\mathrm{rig}} \to \mathbb{A}^{1,\mathrm{rig}}$ has the same property. In particular

$$f \colon S_\alpha = f^{-1}(T_\alpha) \to T_\alpha$$

is a finite étale cover with Galois group I_t, i.e., $T_\alpha \cong I_t \backslash S_\alpha$ as affiniod spaces.

For part (iii) we only sketch the proof of a slightly weaker statement which suffices for all our applications, namely: For each cusp x there exists m_0 such that for all $m_1 \geq m_0$ the set $\rho_\Gamma^{-1}(|C_{x,m_1}|)$ is isomorphic to the $\mathbb{C}_\infty$-valued points of D^*.

Let t be as in the proof of (ii). Then the path in $\mathcal{T}$ from t to ∞ is given by taking for M_m the lattice generated by $(1, y)$ and $(0, \pi_\infty^m)$, $m \leq n$. For $m \leq \mathrm{v}_\infty(y)$, the lattice M_m is generated by the vectors $(1, 0)$ and $(0, \pi_\infty^m)$. Furthermore, by the theorem of Riemann-Roch, there exists an $m_0 \leq \mathrm{v}_\infty(y)$ such that for all $m \leq m_0$ the reduction of $\pi_\infty^{-m} I_{\underline{s}} \cap A_\infty$ modulo $\pi_\infty A_\infty$ equals k_∞.

Let $m_1 \geq m_0$ and let $C_{m_1,\infty}$ be the path given by the M_m for $m \geq m_1$. To identify $\rho_\Gamma^{-1}(|C_{m_1,\infty}|)$ with a punctured closed disc in $\mathbb{C}_\infty$, we define the map

$$z \mapsto e_{I_{\underline{s}}}(z) := z \prod_{i \in I_{\underline{s}} \smallsetminus \{0\}} \left(1 - \frac{z}{i}\right) : \mathbb{C}_\infty \to \mathbb{C}_\infty.$$

One can easily check that $|e_{I_{\underline{s}}}(z)|_\infty = |z|_\infty^{|I_{\underline{s}} \cap \pi_\infty^m A_\infty|} c_m$ for $|\pi_\infty|_\infty^{m-1} > |z|_\infty \geq |\pi_\infty|_\infty^m$ and $m \leq m_0$, where c_m is the constant

$$c_m = \prod_{i \in I_{\underline{s}} \cap \pi_\infty^m A_\infty, i \neq 0} |i|^{-1}.$$

We leave it to the reader to check that the map $z \mapsto e_{I_{\underline{s}}}(z)^{-1}$ induces for each $m \geq m_0$ a biholomorphic map between $\Gamma_{[M_m]} \backslash \mathfrak{U}_{[M_m]}$ and the annulus

$$c_m |\pi_\infty|_\infty^{(m-1/3)|I_{\underline{s}} \cap \pi_\infty^m A_\infty|} \geq |z|_\infty \geq c_m |\pi_\infty|_\infty^{(m-2/3)|I_{\underline{s}} \cap \pi_\infty^m A_\infty|}.$$

Similarly, one can show that if e is the edge connecting $[M_m]$ and $[M_{m+1}]$, then $e_{I_{\underline{s}}}$ induces a biholomorphic map between $\Gamma_e \backslash \mathfrak{U}_e$ and the annulus

$$c_m |\pi_\infty|_\infty^{(m-2/3)|I_{\underline{s}} \cap \pi_\infty^m A_\infty|} \geq |z|_\infty \geq c_{m+1} |\pi_\infty|_\infty^{(m-4/3)|I_{\underline{s}} \cap \pi_\infty^{m-1} A_\infty|}.$$

Cf. [43], §3, for further details. ∎

3.7 Compactifying $\Gamma \backslash \Omega$

We recall the definition of properness for affinoid varieties:

Definition 3.28 *An affinoid subdomain $U := \mathrm{Spm}(B)$ of an affinoid domain $X := \mathrm{Spm}(B')$ is called* relatively compact*, if there exists $f_1, \ldots, f_n$ in B' such that*

(i) the map from the Tate algebra T_n to B' which sends z_i to f_i is surjective, and

(ii) there exists $\varepsilon < 1$ such that U is contained in $\{z \in X : \sup_i |f_i(z)|_\infty \leq \varepsilon\}$.

One writes $U \Subset X$.

A rigid space space X is called proper *(over K_∞) if it has finite admissible affinoid covers $\{\mathfrak{U}_i\}$ and $\{\mathfrak{V}_j\}$ such that for each j there exists an i such that $\mathfrak{V}_j \Subset \mathfrak{U}_i$.*

By Theorem 3.25 we know that for each cusp x there exists $m \geq 0$ such that $\rho_\Gamma^{-1}(|C_{x,m}|)$ is isomorphic to a punctured disc. The rigid analytic space obtained from $\Gamma \backslash \Omega$ by gluing in the punctures for each cusp x is denoted by $\Gamma \backslash \bar{\Omega}$. The notion *cusp* is also used for the finitely many points glued into $\Gamma \backslash \Omega$. As they can be identified with $\Gamma\backslash\mathbb{P}^1(K)$, one defines $\bar{\Omega}(\mathbb{C}_\infty) := \Omega(\mathbb{C}_\infty) \cup \mathbb{P}^1(K)$, so that the $\mathbb{C}_\infty$-valued points of $\Gamma\backslash\bar{\Omega}$ are in bijection with the elements of $\Gamma\backslash(\bar{\Omega}(\mathbb{C}_\infty))$.

Because the graph $\mathbb{Y}$ in Theorem 3.21 is finite, it is simple to see that the rigid space $\Gamma\backslash\bar{\Omega}$, contrary to $\Gamma\backslash\Omega$, has a finite admissible affinoid cover. Furthermore recall that the cover $\mathfrak{U}_\Gamma$ is obtained from the cover $\mathfrak{W}$ of $|\mathcal{T}|$ via ρ^{-1} and quotienting by Γ. If instead of the Čech cover in Definition 3.10, which defines $\mathfrak{W}$, one uses stars and line segments which are slightly larger, and if one applies the assertions of Theorem 3.25 regarding the cusps, the following is rather obvious:

Theorem 3.29 *The space $\Gamma\backslash\bar{\Omega}$ is a proper smooth rigid analytic space over K_∞.*

Remark 3.30 We can now explain the admissibility of $\mathfrak{U}_\Gamma$: Let X be an affinoid subset of $\Gamma\backslash\Omega$. Then X will also be an affinoid subset of $\Gamma\backslash\bar{\Omega}$. For each cusp x let D_x be an affinoid disc around x such that there exists $m_x \in \mathbb{N}$ with $\rho_\Gamma^{-1}(|C_{x,m_x}|) \cong D_x - \{x\}$. Then the set $X \cap D_x$ is an affinoid subset of D_x which does not contain x.

As the sets $\rho_\Gamma^{-1}(|\bar{e}_{x,i}|)$ for $i > m$, in the notation of Theorem 3.21, form an admissible affinoid cover of $D_x \smallsetminus \{x\}$, there exists $m'_x > m_x$ such that $X \cap D_x$ is contained in $\bigcup_{i=m_x}^{m'_x} \rho_\Gamma^{-1}(|\bar{e}_{x,i}|)$. As the subgraph $\mathbb{Y}$ is finite, it follows that X is covered by finitely many affinoids of the cover $\mathfrak{U}_\Gamma$.

Pictorially, one can 'compactify' $\Gamma\backslash\mathcal{T}$ by replacing each cusp Δ_x by the graph $\{v_{x,0}, v_{x,1}, \bar{e}_{x,1}\}$. Call the resulting graph $\Gamma\backslash\bar{\mathcal{T}}$. Then indeed $\Gamma\backslash|\bar{\mathcal{T}}|$ is compact. It is obtained by replacing the non-compact sets $|\Delta_x| \cong \mathbb{R}_{\geq 0}$ of $\Gamma\backslash|\mathcal{T}|$ by compact intervals. According to Theorem 3.25 one can adjust the reduction map ρ_Γ to obtain a map

$$\bar{\rho}_\Gamma \colon \Gamma\backslash\bar{\Omega}(\mathbb{C}_\infty) \longrightarrow \Gamma\backslash|\bar{\mathcal{T}}|.$$

3.8 An affinoid cover of $\Gamma\backslash\bar{\Omega}$

The main point of this subsection is to define a 'good' affinoid cover of $\Gamma\backslash\bar{\Omega}$. We assume for the following that Γ is p'-torsion free. Recall that $\mathcal{T}_\infty$ is the unstable part of $\mathcal{T}$ with respect to Γ and that it is labeled by the rational ends of $\mathcal{T}$.

Let $x \in \Gamma\backslash\mathbb{P}^1(K)$ be a cusp and $[\underline{s}]$ a rational end in the Γ-orbit of x. Let $\mathcal{T}_{\underline{s}}$ denote the connected subtree of $\mathcal{T}_\infty$ whose unique rational cusp is $[\underline{s}]$, and $\Omega_{\underline{s}}$ the union of the affinoid subdomains $\mathfrak{U}_t$ for all simplices t of $\mathcal{T}_{\underline{s}}$. The stabilizer $\Gamma_{\underline{s}}$ acts on $\Omega_{\underline{s}}$. Note that the quotient $\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}$ exists and contains a punctured disc around the cusp s of $\Gamma\backslash\Omega$. (As this holds for any cusp x, there can be only finitely many Γ-orbits of stable simplices!) By $\Gamma_{\underline{s}}\backslash\bar{\Omega}_{\underline{s}}$ we denote the rigid analytic space obtained from $\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}$ by adding the cusp.

Lemma 3.31 *The space $\Gamma_{\underline{s}}\backslash\bar{\Omega}_{\underline{s}}$ is isomorphic to an affinoid subdomain of $\mathbb{P}^{1,\mathrm{rig}}_{K_\infty}$.*

Proof: To see this one only has to follow the proof of Theorem 3.25. Thus, we may assume that $[\underline{s}] = (0:1)$. We define $I_{\underline{s}}$ as in this proof. The proof of parts (ii) and (iii) of Theorem 3.25 show that the image in $\mathbb{P}^{1,\mathrm{rig}}_{K_\infty}$ of $\bigcup_{t\in\mathcal{T}_{\underline{s}}} \mathfrak{U}_t$ under $e_{I_{\underline{s}}}$ consists of a punctured disc together with a finite union of affinoid sets. If we add in the puncture, the image will consist of the complement in $\mathbb{P}^{1,\mathrm{rig}}_{K_\infty}$ of finitely many disjoint open discs. Thus it is an affinoid as asserted. ∎

For an arithmetic subgroup Γ which is p'-torsion free, we define the following finite admissible cover $\bar{\mathfrak{U}}_\Gamma$ of $\Gamma\backslash\bar{\Omega}$: For each cusp x choose a rational end $[\underline{s}_x]$ representing it. For each stable Γ-orbit x' in the set of simplices of $\Gamma\backslash(\mathcal{T} \smallsetminus \mathcal{T}_\infty)$ choose a representing simplex $t_{x'}$. Define $\mathfrak{U}_x$ to be the affinoid $\Gamma_{\underline{s}_x}\backslash\bar{\Omega}_{\underline{s}_x}$, and define $\mathfrak{U}_{x'}$ to be the affinoid $\Gamma\backslash(\Gamma\mathfrak{U}_{t_{x'}}) \cong \mathfrak{U}_{t_{x'}}$. Let

$$\bar{\mathfrak{U}}_\Gamma := \{\mathfrak{U}_x : x \text{ a cusp of } \Gamma\backslash\Omega\} \cup \{\mathfrak{U}_{x'} : x' \text{ a simplex of } \Gamma\backslash(\mathcal{T} \smallsetminus \mathcal{T}_\infty)\}.$$

Proposition 3.32 *The collection $\bar{\mathfrak{U}}_\Gamma$ of affinoid subdomains of $\Gamma\backslash\bar{\Omega}$ is a finite admissible cover.*

The proof is obvious as the affinoids in $\bar{\mathfrak{U}}_\Gamma$ cover $\Gamma\backslash\bar{\Omega}$ by their very definition.

4 Drinfeld modular varieties, the analytic side

Our definition of rank r Drinfeld-A-modules with some level structure carries over verbatim from the algebraic to the rigid analytic setting over K_∞ or $\mathbb{C}_\infty$, and hence so does our discussion of Drinfeld modular varieties and their existence, cf. also [10], [15], [58] and [57]. As a result, one obtains rigid analytic moduli spaces $\mathfrak{M}_{\mathfrak{n}}^{\mathrm{rig}}$ and $\mathfrak{M}_{\mathcal{K}}^{\mathrm{rig}}$ as the rigidification of the corresponding algebraic spaces. In this section we will describe an explicit model for these spaces as disjoint unions of quotients $\Gamma_i\backslash\Omega$. Another issue will be their compactification. None of this is original and most of it can be found in the above sources.

Throughout this section, we fix a complete valued field $L \subset \mathbb{C}_\infty$ containing the valued field $(K_\infty, |\,.\,|)$. Also we regard $\Omega(\mathbb{C}_\infty)$ as a subset of $\mathbb{A}^1(\mathbb{C}_\infty)$, i.e., we identify the point $\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) \in \mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus \{\infty\}$ with $z_1/z_2 \in \mathbb{C}_\infty$. Again $\mathcal{K}$ denotes an open subgroup of $\mathrm{GL}_2(\hat{A})$.

4.1 Lattices

It is well known that over $\mathbb{C}_\infty$ there is a bijection between rank r Drinfeld-A-modules and projective A-lattices of rank r in $\mathbb{C}_\infty$, cf. [8], Sect. 2.2. In the following we will introduce a generalization of the above to general rigid analytic varieties. This will in general *not* give a bijection, but will be useful in the construction of Drinfeld-modules over a general rigid analytic base.

Let B be an affinoid algebra over L and let $|\_\_|_B$ denote any residue norm. In particular, B is a Banach algebra with respect to $|\_\_|_B$. Also, let ι_B be the composition $A \stackrel{\iota_{K_\infty}}{\to} K_\infty \to L \to B$. A subset Λ of B will be called *a discrete A-lattice of rank r*, if the following hold:

(i) Λ is a projective A-module of rank r via ι_B,

(ii) the elements of $\Lambda \smallsetminus \{0\}$ are units and

(iii) for all $c \in \mathbb{R}_{>0}$ the set $\{x \in \Lambda \smallsetminus \{0\} : |x^{-1}|_B \geq c\}$ is finite.

Definition 4.1 *Let S be a rigid analytic space over K_∞. A* local system $(\mathcal{L}, \underline{\Lambda}, s)$ of A-lattices of rank r on S, *or short an* A-lattice of rank r over S, *consists of the following data:*

(i) A (rigid analytic) line bundle $\mathcal{L}$ on S.

(ii) A sheaf $\underline{\Lambda}$ of locally free A-lattices of rank r over the rigid analytic site.

(iii) A monomorphism $s\colon \underline{\Lambda} \to \mathcal{L}$ of sheaves of A-modules such that locally on the rigid analytic site $\underline{\Lambda}$ is a discrete A-lattice of rank r in $\mathcal{L}$.

Note that for $S = \mathrm{Spm}(\mathbb{C}_\infty)$ our definition agrees with the usual one, while if L is not algebraically closed, it is more restrictive, cf. [23], Thm. 4.6.9.

Let $\mathcal{M}^{r,\mathrm{rig}}$ denote the fibered category that associates to each rigid analytic space S, the set of all Drinfeld-A-modules of rank r over S. Similarly, one defines the fibered category of all A-lattices of rank r as $\mathcal{N}^{r,\mathrm{rig}}$. One has the following basic result modeled after [10], Prop.3.1.

Proposition 4.2 *There exists a natural transformation $\Psi^r : \mathcal{N}^{r,\mathrm{rig}} \to \mathcal{M}^{r,\mathrm{rig}}$. For $S = \mathrm{Spm}(\mathbb{C}_\infty)$ it is an equivalence.*

PROOF: We will only provide the construction in the case that $S = \mathrm{Spm}(B)$ is an affinoid, that $H^0(S, \mathcal{L}) = B$ and that $\Lambda := H^0(S, \underline{\Lambda})$ is a rank r lattice. The general construction can be obtained from the local one by gluing.

Define
$$e_\Lambda \colon B \to B : z \mapsto z \prod_{\alpha \in \Lambda, \alpha \neq 0} (1 - \frac{z}{\alpha}).$$

The argument of [10], Prop.3.1, shows that the following hold:

(i) The expression for e_Λ is a convergent power series which defines an analytic function on B.

(ii) e_Λ is k-linear, i.e., for all $\alpha \in k$ and all $x, y \in B$, $e_\Lambda(\alpha x + y) = \alpha e_\Lambda(x) + e_\Lambda(y)$.

(iii) For all $a \in A$, there exists a unique k-linear polynomial $\varphi_a(x) \in B[x]$, namely
$$ax \cdot \prod_{\gamma \in a^{-1}\Lambda/\Lambda \smallsetminus \{0\}} \Big(1 - \frac{x}{e_\Lambda(\gamma)}\Big),$$
such that the following diagram commutes:
$$\begin{tikzcd} 0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "a\cdot\_"] & B \arrow[r, "e_\Lambda"] \arrow[d, "a\cdot\_"] & B \arrow[d, "\varphi_a"] \\ 0 \arrow[r] & \Lambda \arrow[r] & B \arrow[r, "e_\Lambda"] & B. \end{tikzcd} \tag{12}$$

The uniqueness of φ_a might need some clarification, as the horizontal maps on the right in the above diagram need not be surjective. First note that by [5], Thm. 5.2.6.3, the ring B is Jacobson, and hence that the intersection of all ideals I of B such that B/I has finite length is zero. Therefore it suffices to prove the uniqueness of φ_a after completing B at any maximal ideal. Also note, that we may base change to $\mathbb{C}_\infty$. In this situation, the assertion is a consequence of the following lemma, which we state without proof.

Lemma 4.3 *Let $\mathfrak{m}$ be a maximal ideal of B and assume that $L = \mathbb{C}_\infty$. Then after completing B with respect to $\mathfrak{m}$, the map e_Λ becomes surjective.*

We continue with the proof of Proposition 4.2. By k-linearity, the coefficient of x^j in φ_a can only be non-zero if j is a power of q. The element $\varphi_a \in B\{\tau\}$ is defined by replacing the term x^{q^i} in $\varphi_a(x)$ by τ^i. Given the above properties, it is easy to check that $A \to B\{\tau\} : a \to \varphi_a$ defines a Drinfeld A-module on B. Finally, the assertion on the case $\mathbb{C}_\infty$ is classical and can be found in [10], Prop. 3.1. ∎

Remark 4.4 One can in fact show that $\mathcal{M}^{r,\mathrm{rig}} \xrightarrow{\Psi^r} \mathcal{N}_{\mathfrak{n}}^{r,\mathrm{rig}}$ is the minimal hull of $\mathcal{M}^{r,\mathrm{rig}}$ which is closed under faithfully flat (or simply étale) descent.

Next we want to define level structures for lattices. For any abelian group G, we define $\underline{G}_S$ as the constant sheaf with stalk G on the rigid site of S. This is analogous to the definition above Proposition 1.5, and so there should arise no confusion.

Definition 4.5 *An isomorphism $\tilde{\psi} : \underline{(\mathfrak{n}^{-1}/A)^r_S} \to \mathfrak{n}^{-1}\underline{\Lambda}/\underline{\Lambda}$ is called a* level $\mathfrak{n}$-structure *of $(\mathcal{L}, \underline{\Lambda}, s)$.*

For $r = 2$ and $\mathcal{K}$ an open subgroup of $\mathrm{GL}_2(\hat{A})$ of conductor $\mathfrak{n}$, we make the following definition.

Definition 4.6 *Two level* $\mathfrak{n}$*-structures* $\tilde{\psi}, \tilde{\psi}'$ *are called* $\mathcal{K}$-equivalent *if there exists* $g \in \mathcal{K}$ *such that* $\tilde{\psi}' = \tilde{\psi} g \pmod{\mathfrak{n}}$*. A* level $\mathcal{K}$-structure $[\tilde{\psi}]$ *is a* $\mathcal{K}$*-equivalence class of level* $\mathfrak{n}$*-structures.*

Let $\mathcal{M}_{\mathfrak{n}}^{r,\mathrm{rig}}$ denote the fibered category that associates to each rigid analytic space S, the set of all Drinfeld-A-modules of rank r over S together with a level $\mathfrak{n}$-structure. Let $\mathcal{N}_{\mathfrak{n}}^{r,\mathrm{rig}}$ denote the corresponding category for rank r lattices. Analogously, one defines $\mathcal{M}_{\mathcal{K}}^{\mathrm{rig}}$ and $\mathcal{N}_{\mathcal{K}}^{\mathrm{rig}}$, for an open subgroup $\mathcal{K} \subset \mathrm{GL}_2(\hat{A})$.

Proposition 4.7 *The natural transformation* Ψ^r *of Proposition 4.2 extends to a natural transformation* $\Psi_{\mathfrak{n}}^r : \mathcal{N}_{\mathfrak{n}}^{r,\mathrm{rig}} \to \mathcal{M}_{\mathfrak{n}}^{r,\mathrm{rig}}$*. The image of* $\Psi_{\mathfrak{n}}^r$ *is the set of pairs* $(\underline{\varphi}, \psi)$ *of Drinfeld-modules* $\underline{\varphi}$ *with a level structure* ψ*, such that* $\underline{\varphi}$ *is in the image of* Ψ_r*. The analogous assertion holds for level* $\mathcal{K}$*-structures.*

PROOF: We only give the proof for level $\mathfrak{n}$-structures and in the case that $S = \mathrm{Spm}(B)$, $H^0(S, \mathcal{L}) = B$ and $\Lambda := H^0(S, \underline{\Lambda})$ is an A-lattice of rank r. Let $\tilde{\psi} \colon (\mathfrak{n}^{-1}/A)^r \xrightarrow{\cong} \mathfrak{n}^{-1}\Lambda/\Lambda$ be a level $\mathfrak{n}$-structure.

Composing e_Λ with $\tilde{\psi}$ gives a map $\psi \colon (\mathfrak{n}^{-1}/A)^r \to B$ which satisfies $\underline{\varphi}_a \psi = 0$ for all $a \in \mathfrak{n}$. Therefore ψ takes its values in $\underline{\varphi}[\mathfrak{n}]$. Comparing cardinalities of $(A/\mathfrak{n})^r$ and the image of ψ composed with $B \to B/\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of B, yields that $\psi \colon (\mathfrak{n}^{-1}/A)^r \to \underline{\varphi}[\mathfrak{n}]$ is an isomorphism. This is the desired extension of Ψ_r.

Finally, let $\underline{\varphi}$ be a Drinfeld-module which is the image under Ψ^r of a local system $\underline{\Lambda}$ of rank r, and let ψ be a level $\mathfrak{n}$-structure on $\underline{\varphi}$. The above argument can be reversed to construct a corresponding level $\mathfrak{n}$-structure on $\underline{\Lambda}$. ∎

4.2 The moduli problem

Theorem 4.8 *Let* $\mathfrak{n}$ *be a proper non-zero ideal of* A*. Then* $\mathcal{M}_{\mathfrak{n}}^{r,\mathrm{rig}}$ *is representable by a smooth rigid analytic space over* L*. It is naturally isomorphic to the rigidification of the algebraic moduli space of Drinfeld-*A*-modules of rank* r *over* L *with a level* $\mathfrak{n}$*-structure.*

Furthermore, if $\mathcal{K}$ *is an admissible open subgroup of* $\mathrm{GL}_2(\hat{A})$*, then the above holds for* $\mathcal{M}_{\mathcal{K}}^{\mathrm{rig}}$*, too.*

We will only give names to the moduli spaces in the rank 2 case, and denote them by $\mathfrak{M}_{\mathfrak{n}}^{\mathrm{rig}}$ and $\mathfrak{M}_{\mathcal{K}}^{\mathrm{rig}}$, respectively. If we want to indicate the base field L, we write $\mathfrak{M}_{\mathfrak{n},L}^{\mathrm{rig}}$ and $\mathfrak{M}_{\mathcal{K},L}^{\mathrm{rig}}$, respectively.

PROOF: The proof is rather a triviality, once the algebraic proof is established. We give it for $r = 2$ and level $\mathfrak{n}$-structures. Let $\mathfrak{M}_{\mathfrak{n}}^{\mathrm{rig}}$ denote the rigidification of $\mathfrak{M}_{\mathfrak{n},L}$, the moduli space for rank 2 Drinfeld-modules over L with a level $\mathfrak{n}$-structure. Let $(\underline{\varphi}_{\mathfrak{n}}^{\mathrm{rig}}, \psi_{\mathfrak{n}}^{\mathrm{rig}})$ be the pair induced from the universal Drinfeld-module on $\mathfrak{M}_{\mathfrak{n},L}$ together with its universal level structure. We will directly show that $(\mathfrak{M}_{\mathfrak{n}}^{\mathrm{rig}}, \underline{\varphi}_{\mathfrak{n}}^{\mathrm{rig}}, \psi_{\mathfrak{n}}^{\mathrm{rig}})$ represents $\mathcal{M}_{\mathfrak{n}}^{2,\mathrm{rig}}$.

Let $(\underline{\tilde{\varphi}}, \tilde{\psi})$ be a Drinfeld-module over the rigid space S. By the universality of $\mathfrak{M}_{\mathfrak{n},L}$, it arises from $(\underline{\varphi}_{\mathfrak{n}}, \varphi_{\mathfrak{n}})$ on $\mathfrak{M}_{\mathfrak{n},L}$ via a unique map from S to $\mathfrak{M}_{\mathfrak{n},L}$. It is simple to check that this map induces a morphism $S \to \mathfrak{M}_{\mathfrak{n}}^{\mathrm{rig}}$, under which $(\underline{\varphi}, \psi)$ arises from $(\underline{\varphi}_{\mathfrak{n}}^{\mathrm{rig}}, \psi_{\mathfrak{n}}^{\mathrm{rig}})$, and that there is a unique such morphism. ∎

4.3 Analytic uniformization

Similar to the classical case when describing modular curves over the complex numbers as quotients of the upper half plane, which is simple and very explicit, one can obtain an explicit description of $\mathfrak{M}_{\mathcal{K}}^{\mathrm{rig}}$. A slight complication in the Drinfeld modular case arises as A may have class number larger than one, and therefore we will usually work in an adelic setting. We follow closely [8] and will use the functor $\Psi_{\mathcal{K}}$.

To get started, suppose that we want to describe $\mathcal{M}_A^{\mathrm{rig}}(\mathbb{C}_\infty)$. By Proposition 4.2, we need to describe discrete rank 2 lattices over A inside $\mathbb{C}_\infty$ modulo dilatations. This task can be decomposed into three steps. First describe sublattices Λ of K^2. Via $K^2 \to K_\infty^2 = V_\infty$ this gives rank 2 sublattices of K_∞. Second, describe the monomorphisms $V_\infty \to \mathbb{C}_\infty$ up to dilatations — they can be identified with points $z \in \Omega(\mathbb{C}_\infty) \subset \mathbb{C}_\infty$. Third, determine when different pairs (Λ, z) are isomorphic. Once one has added also level structures to the above, this gives a 'recipe' to construct $\mathfrak{M}_{\mathcal{K}}^{\mathrm{rig}}$.

We first give the description of the lattices (which works with obvious modifications for any rank). Note that K^2, as well as $\hat{A}^2$ sit canonically inside $(\mathbb{A}^f)^2$ which is viewed as a module of row vectors. For $g \in \mathrm{GL}_2(\mathbb{A}^f)$, define $\Lambda_g := \hat{A}^2 g^{-1} \cap K^2 \subset (\mathbb{A}^f)^2$. This is an A-lattice of rank 2.

Suppose we are also given a non-zero ideal $\mathfrak{n}$ of A. Then the element g gives canonically a level $\mathfrak{n}$-structure $\tilde{\psi}_g$ on Λ_g, via

$$(\mathfrak{n}^{-1}/A)^2 \xrightarrow{\cong} (\mathfrak{n}^{-1}\hat{A}^2 g^{-1}/\hat{A}^2 g^{-1}) \xleftarrow{\cong} \mathfrak{n}^{-1}\Lambda_g/\Lambda_g,$$

where the left isomorphism is multiplication on the right by g^{-1}. For the right isomorphism, one needs to show that $\hat{A}^2/\Lambda_g$ is a divisible A-module.

The following lemma, which is implicitly contained in op. cit., we state without proof.

Lemma 4.9 *The set $\{(\Lambda_g, \tilde{\psi}_g) : g \in \mathrm{GL}_2(\mathbb{A}^f)\}$ is the set of all A-sublattices with a level $\mathfrak{n}$-structure in K^2 of rank 2. Two pairs $(\Lambda_g, \tilde{\psi}_g)$ and $(\Lambda_{g'}, \tilde{\psi}_{g'})$ are isomorphic, if and only if $g' \in g\mathcal{K}$.*

We now consider the product $\Omega \times (\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}) =: \Omega_{\mathcal{K}}$, which is an infinite sum of identical copies of Ω. We denote the copy corresponding to $g\mathcal{K}$ by Ω_g. Define $\mathcal{L} := \mathcal{O}_{\Omega_{\mathcal{K}}} = \mathcal{O}_\Omega \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ and $\underline{\Lambda}$ as the sheaf of A-lattices $\coprod_{g \in \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}} (\Lambda_g)_{\Omega_g}$, where $(\Lambda_g)_{\Omega_g}$ is the constant sheaf on Ω_g with stalk Λ_g.

For $g \in \mathrm{GL}_2(\mathbb{A}_f)$ and $y = (y_0, y_1) \in \Lambda_g \subset K^2$, define a section $s_{g,y}$ in $\mathcal{O}_{\Omega_g}$ by

$$s_{g,y}(z) := y_0 z + y_1.$$

where $z \in \Omega \subset \mathbb{C}_\infty$. For fixed g and varying y, the collection of all these sections gives rise to a map of rigid analytic sheaves

$$(\Lambda_g)_{\Omega_g} \hookrightarrow \mathcal{O}_{\Omega_g}.$$

Let $\{y_g\}_{g \in \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}}$ be any choice of elements $y_g \in \Lambda_g$. Then we denote by $s_{\{y_g\}}$ the section of $\Omega_{\mathcal{K}}$ which on Ω_g is given by s_{g,y_g}. Via the maps s_{y_g} one obtains a monomorphism of rigid analytic sheaves of A-modules $s \colon \underline{\Lambda} \hookrightarrow \mathcal{L}$. Furthermore, the maps $\tilde{\psi}_g$ patch to give rise to an isomorphism $(\mathfrak{n}^{-1}/A)^2_{\Omega_{\mathcal{K}}} \to \mathfrak{n}^{-1}\underline{\Lambda}/\underline{\Lambda}$. Using the affinoids $\mathfrak{U}_t$, $t \in T$, the reader may verify the following result.

Proposition 4.10 *The quadruple $(\mathcal{L}, \underline{\Lambda}, s, \psi)$ is a local system of A-lattices of rank 2 with a level $\mathfrak{n}$-structure on $\Omega_{\mathcal{K}}$.*

Next we describe an action of $\mathrm{GL}_2(K)$ on $(\mathcal{L}, \underline{\Lambda}, s, \psi)$ on the left so that over $\mathbb{C}_\infty$ the quotient of $\Omega_\mathcal{K}$ modulo this action parametrizes A-lattices of rank 2 in $\mathbb{C}_\infty$ with a level $\mathcal{K}$-structure modulo dilatations.

Let γ be in $\mathrm{GL}_2(K)$. We let γ act on the row vectors in $(\mathbb{A}^f)^2$ by having its inverse act on the right, so that in particular one has $\gamma \circ \Lambda_g = \Lambda_g \gamma^{-1} = \Lambda_{\gamma g}$. The element γ maps $z \in \Omega$ to γz. Furthermore, $z \in \Omega$ gives rise to a monomorphism $V_\infty \to \mathbb{C}_\infty$, up to a dilatation, by mapping $(y_0, y_1) \in V_\infty$ to $zy_0 + y_1$. It is simple to check that the dilatation class determined by Λ_g and z is the same as that determined by $\gamma \circ \Lambda_g$ and γz. This leads to the definitions of the following actions of $\mathrm{GL}_2(K)$, which we describe for the element γ, where $\gamma^{-1} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$:

$$\begin{array}{rcl} (z, g\mathcal{K}) \in \Omega_\mathcal{K} & \mapsto & (\gamma z, \gamma g \mathcal{K}) \\ f \in \Gamma(\Omega_\mathcal{K}, \mathcal{L}) & \mapsto & (c'z + d') f \circ \gamma^{-1} \\ (y, g\mathcal{K}) \in \underline{\Lambda} & \mapsto & (y\gamma^{-1}, \gamma g \mathcal{K}) \quad y \in \Lambda_g \\ \psi \colon (\mathfrak{n}^{-1}/A)^2 \to \mathfrak{n}^{-1}\Lambda_g/\Lambda_g & \mapsto & \gamma \circ \psi \colon (\mathfrak{n}^{-1}/A)^2 \to \mathfrak{n}^{-1}\Lambda_{\gamma g}/\Lambda_{\gamma g} \end{array}$$

In particular, one verifies from the above definitions that $\gamma \circ s_{\{y_g\}} = s_{\{y_{\gamma^{-1}g}\gamma^{-1}\}}$. Via $\Psi_\mathcal{K}$, this induces an action of $\mathrm{GL}_2(K)$ on $\Omega_\mathcal{K}$ together with the Drinfeld-module of rank 2 with a level $\mathcal{K}$-structure, $(\underline{\varphi}(\mathcal{K}), \psi(\mathcal{K})) := \Psi_\mathcal{K}(\mathcal{L}, \underline{\Lambda}, s, \psi)$. In [10], Prop. 6.6, it is shown that the map which sends a point of $\mathrm{GL}_2(K)\backslash\Omega_\mathcal{K}(\mathbb{C}_\infty)$ to the corresponding $\mathrm{GL}_2(K)$-equivalence class of $(\underline{\varphi}(\mathcal{K}), \psi(\mathcal{K}))$ induces a bijection

$$\mathrm{GL}_2(K)\backslash\Omega_\mathcal{K}(\mathbb{C}_\infty) \xrightarrow{\cong} \mathcal{M}_\mathcal{K}(\mathbb{C}_\infty). \tag{13}$$

Remark 4.11 The local system $(\mathcal{L}, \underline{\Lambda}, s, \psi)$ does not descend to $\mathrm{GL}_2(K)\backslash\Omega_\mathcal{K}$, because for $y_0 \neq 0$, the section $s_{g,(y_0,0)} : z \mapsto y_0 z$ on any Ω_g is not invariant under $\Gamma_g = \mathrm{GL}_2(K) \cap g^{-1}\mathcal{K}g$. So while there is a universal Drinfeld-module on the quotient, there is no local system on the quotient that could define it. Such a system only exists on the stable part of $\Omega_\mathcal{K}$.

We now want to give a local description of the bijection (13). This will show that $\mathrm{GL}_2(K)\backslash\Omega_\mathcal{K}$ is a meaningful definition of a rigid analytic space over K_∞ which carries a universal Drinfeld-module of rank 2 together with a level $\mathcal{K}$-structure.

Lemma 4.12 *For any open subgroup $\mathcal{K} \subset \mathrm{GL}_2(\hat{A})$ the determinant map induces a bijection of double cosets*

$$\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K} \xrightarrow{\cong} K^*\backslash(\mathbb{A}^f)^*/\det(\mathcal{K}) =: \mathrm{Cl}_\mathcal{K},$$

where $\det(\mathcal{K})$ *denotes the image of* $\mathcal{K}$ *in* $\hat{A}^*$ *under* $\det$.

Proof: Let $\mathcal{K}' := \mathcal{K} \cap \mathrm{SL}_2(\mathbb{A}^f)$. The strong approximation theorem, [28], §14.3, asserts that $\mathrm{SL}_2(K)\mathcal{K}' = \mathrm{SL}_2(\mathbb{A}^f)$. An application of the snake lemma to the following diagram yields the desired result:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{SL}_2(K)\mathcal{K}' \arrow[r] \arrow[d, equal] & \mathrm{GL}_2(K)\mathcal{K} \arrow[r, "\det"] \arrow[d] & K^* \det(\mathcal{K}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{SL}_2(\mathbb{A}^f) \arrow[r] & \mathrm{GL}_2(\mathbb{A}^f) \arrow[r, "\det"] & (\mathbb{A}^f)^* \arrow[r] & 0. \end{tikzcd}$$ ∎

Remark 4.13 The above bijection identifies $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ with the ray class group $\mathrm{Cl}_{\mathcal{K}}$ of the maximal abelian extension of K of conductor $\det\mathcal{K}$ which is totally split above ∞. In particular $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ is finite.

Let $\{t_\nu\}_\nu \subset (\mathbb{A}^f)^*$ be a set of representatives of $K^*\backslash(\mathbb{A}^f)^*/\det(\mathcal{K})$ and define

$$x_\nu := \begin{pmatrix} 1 & 0 \\ 0 & t_\nu \end{pmatrix} \in \mathrm{GL}_2(\mathbb{A}^f). \tag{14}$$

By the above lemma, the x_ν form a set of representatives of $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$. Correspondingly, we set $\Gamma_\nu := \mathrm{GL}_2(K)\cap x_\nu\mathcal{K}x_\nu^{-1}$. Thus for $\gamma,\gamma' \in \mathrm{GL}_2(K)$ and $g, g' \in \mathcal{K}$ one has $\gamma x_\nu g = \gamma' x_\nu g' \iff \gamma^{-1}\gamma' \in \Gamma_\nu$.

Lemma 4.14 *The groups Γ_ν are arithmetic subgroups of $\mathrm{GL}_2(K)$. If $\mathcal{K}$ is admissible, then the Γ_ν are p'-torsion free.*

Proof: It is not hard to see that there exists a non-zero ideal $\mathfrak{n}$ of A such that $\mathcal{K}(\mathfrak{n}) \subset x_\nu\mathcal{K}x_\nu^{-1}$. By compact-openness of the groups involved, the inclusion is of finite index. Thus

$$\Gamma(\mathfrak{n}) = \mathcal{K}(\mathfrak{n}) \cap \mathrm{GL}_2(K) \subset x_\nu\mathcal{K}x_\nu^{-1} \cap \mathrm{GL}_2(K) = \Gamma_\nu$$

is an inclusion of finite index. This proves the first assertion.

For the second suppose that $\gamma \in \Gamma_\nu \subset \mathrm{GL}_2(K)$ is of finite order prime to p. As k is the field of constants of K, the minimal polynomial of γ is defined over k. Thus the rational canonical form of γ over K lies in $\mathrm{GL}_2(k)$. Hence a $\mathrm{GL}_2(K)$-conjugate of γ lies in $\mathrm{GL}_2(k)$. As $\mathrm{GL}_2(K) \subset \mathrm{GL}_2(\mathbb{A}^f)$, the element γ must be of p-power order by the definition of admissibility. It follows that γ is trivial. ∎

By the results of the previous section, the quotient $\Gamma_\nu\backslash\Omega$ exists as a rigid space for any $\nu \in \mathrm{Cl}_{\mathcal{K}}$. Hence the isomorphism

$$\mathrm{GL}_2(K)\backslash\Omega_{\mathcal{K}} \cong \coprod_{\nu\in\mathrm{Cl}_{\mathcal{K}}} \Gamma_\nu\backslash\Omega$$

shows that the left hand side has a meaningful description as a rigid analytic space. As $\mathrm{GL}_2(K)$ acts freely on $(\underline{\varphi}(\mathcal{K}), \psi(\mathcal{K}))$, it follows that it induces a Drinfeld-module of rank 2 with a level $\mathcal{K}$-structure $\mathrm{GL}_2(K)\backslash(\underline{\varphi}(\mathcal{K}), \psi(\mathcal{K}))$ on the quotient space $\mathrm{GL}_2(K)\backslash\Omega_{\mathcal{K}}$. The following is from [10], Prop. 6.6:

Theorem 4.15 *Let $\xi : \mathrm{GL}_2(K)\backslash\Omega_{\mathcal{K}} \longrightarrow \mathfrak{M}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$ be the canonical map that arises from the Drinfeld module $\mathrm{GL}_2(K)\backslash(\underline{\varphi}(\mathcal{K}), \psi(\mathcal{K}))$ via the universality of $\mathfrak{M}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$. Then ξ is an isomorphism.*

For the proof, one notes that ξ is a map between two smooth rigid analytic spaces over K_∞ of dimension one, that ξ is an isomorphism on $\mathbb{C}_\infty$-valued points, and one shows that the map is an isomorphism on tangent spaces.

4.4 Compactification

We have two natural compactifications of $\mathrm{GL}_2(K)\backslash\Omega_{\mathcal{K}} \xrightarrow{\cong} \mathfrak{M}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$ for an admissible $\mathcal{K}$: On the one hand, Theorem 3.29 yields a smooth compactification of $\coprod_{\nu\in\mathrm{Cl}_{\mathcal{K}}} \Gamma_\nu\backslash\Omega$ by compactifying each component. On the other, we are given in Theorem 2.14 Drinfeld's smooth compactification of $\mathfrak{M}_{\mathcal{K}}$ over $\mathrm{Spec}\,A(\mathfrak{n})$, where $\mathfrak{n}$ is a conductor of $\mathcal{K}$. After base change to $\mathrm{Spec}\,K_\infty$, rigidifying yields $\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$ as a second compactification of $\mathfrak{M}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$.

Theorem 4.16 *The above two compactifications of* $\mathfrak{M}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$ *are canonically isomorphic, i.e., there exists a unique extension*

$$\bar{\xi}\colon \coprod_{\nu\in\mathrm{Cl}_\mathcal{K}} \Gamma_\nu\backslash\bar{\Omega} \xrightarrow{\cong} \overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty} \tag{15}$$

of the isomorphism $\xi\colon \mathrm{GL}_2(K)\backslash\Omega_\mathcal{K} \longrightarrow \mathfrak{M}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$.

Proof: Again for lack of a suitable reference, we sketch a proof. We note first that by a simple Galois descent argument, it suffices to give the proof in the case $\mathcal{K} = \mathcal{K}(\mathfrak{n})$. So, let I be the finite set $\overline{\mathfrak{M}}_{\mathfrak{n},K_\infty} \smallsetminus \mathfrak{M}_{\mathfrak{n},K_\infty}$, and let $\{c_i\}_{i\in I}$ denote the corresponding points. Clearly the c_i are also the points of $\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathfrak{n},K_\infty} \smallsetminus \mathfrak{M}^{\mathrm{rig}}_{\mathfrak{n},K_\infty}$.

For each c_i, we choose a rational function f_i on $\overline{\mathfrak{M}}_{\mathfrak{n},K_\infty}$ whose only pole is at c_i and define $\mathfrak{U}_i = \{z : |f_i(z)| \geq r_i\}$ for some sufficiently large $r_i \in \mathbb{R}$, such that the $\mathfrak{U}_i$ are pairwise disjoint. Furthermore, if $\mathbb{Y}_\nu$ denotes the connected subgraph of $\Gamma_\nu\backslash\mathcal{T}$ from Theorem 3.21, and ρ_ν the reduction map $\Gamma_\nu\backslash\Omega \to \Gamma_\nu\backslash\mathcal{T}$, then we also assume that all the $\mathfrak{U}_i$ are disjoint from the union X_0 of the $\rho_\nu^{-1}(\mathbb{Y}_\nu)$, viewed as a finite union of affinoid subdomains of $\mathfrak{M}^{\mathrm{rig}}_{\mathfrak{n},K_\infty}$ via ξ. Finally, note that $\mathfrak{U}_i$ is isomorphic to a closed disk because the map

$$\mathfrak{U}_i \to D_i := \{w : |w| \geq r_i\} : z \mapsto f_i(z)$$

is an m_i-fold cover of D_i only ramfied at c_i with ramification degree m_i, the order of pole of f_i at c_i (for sufficiently small r_i).

Let $j \in J$ be an enumeration of the cusps c'_j of $\coprod_\nu \Gamma_\nu\backslash\Omega$. Write $i \in \nu$ if c_i is a cusp of $\Gamma_\nu\backslash\Omega$. We now construct a bijection between the c_i and the c'_j: Denote by $\mathfrak{U}_i^c$ the affinoid $\{z : |f_i(z)| \leq r_i\}$. As the intersection of affinoids is again an affinoid, the set $U^c := \bigcup_\nu \bigcap_{i\in\nu} \mathfrak{U}_i^c$ is a disjoint union of affinoids. By the construction of the $\mathfrak{U}_i$, the affinoid U^c lies inside $\coprod_\nu \Gamma_\nu\backslash\Omega$ and contains X_0. Hence for each j there are discs $D'_j \subset D_j \subset \coprod_\nu \Gamma_\nu\backslash\bar{\Omega}$ around the cusp c'_j, such that the D_j are disjoint and

$$\bigcup (D_j \smallsetminus \{c'_j\}) \supset \bigcup (\mathfrak{U}_i \smallsetminus \{c_i\}) \supset \bigcup (D'_j \smallsetminus \{c'_j\}).$$

As the $\mathfrak{U}_i$ are connected and the D_j are disjoint and connected, each $\mathfrak{U}_i \smallsetminus \{c_i\}$ lies inside at most one $D_j \smallsetminus \{c'_j\}$. For the same reason, each $D'_j \smallsetminus \{c'_j\}$ lies in at most one $\mathfrak{U}_i \smallsetminus \{c_i\}$. So from now on, we assume that $I = J$ and $D'_i \smallsetminus \{c'_i\} \hookrightarrow \mathfrak{U}_i \smallsetminus \{c_i\} \hookrightarrow D_i \smallsetminus \{c'_i\}$. We claim that there exist unique rigid analytic maps $D'_i \hookrightarrow \mathfrak{U}_i \hookrightarrow D_i$ extending the above inclusions. This will complete the proof of the theorem.

As the cusps c'_i are defined over K_∞, after a base change to the field of definition of c_i, the discs D'_i and D_i will remain connected. As we may choose $\mathfrak{U}_i$ arbitrarily small, this shows that the field of definition of c_i must be purely inseparable over K_∞, and hence equal to K_∞ by smoothness of $\overline{\mathfrak{M}}_{\mathfrak{n},K_\infty}$ over K_∞.

Thus each of $D'_i\smallsetminus\{c'_i\}$, $\mathfrak{U}_i\smallsetminus\{c_i\}$ and $D_i\smallsetminus\{c'_i\}$ is isomorphic to $D^* := \{z : 0 < |z| \leq 1\}$, and one may assume that $D'_i \smallsetminus \{c'_i\} \hookrightarrow \mathfrak{U}_i \smallsetminus \{c_i\} \hookrightarrow D_i \smallsetminus \{c'_i\}$ is given by explicit maps $D^* \overset{\alpha}{\hookrightarrow} D^* \hookrightarrow D^*$, where the composite map is multiplication by a non-zero constant of norm at most 1. Based on the characterization of non-vanishing functions on an annulus, and the injectivity of the given maps, one can show that $\alpha(z) = az(1+\sum_{n>0} a_n z^n)$ where a and the a_n are in K_∞, the sequence $|a_n|$ converges to zero and all a_n are of norm less than one. Obviously, this map extends in a unique way to a biholomorphic function in a neighborhood of zero. ∎

In Proposition 3.32 we exhibited an affinoid cover $\bar{\mathfrak{U}}_\Gamma$ of $\Gamma\backslash\bar{\Omega}$ for any arithmetic Γ. Via the isomorphism (15) these local affinoid covers can be combined to give a finite affinoid cover $\bar{\mathfrak{U}}_{\mathcal{K}}$ of $\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$. One can check that this cover is independent of any choices used in the isomorphism.

Definition 4.17 *The cover $\bar{\mathfrak{U}}_{\mathcal{K}}$ is called the* standard affinoid cover *of $\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$.*

We record another corollary of Theorem 4.16.

Corollary 4.18 *For any arithmetic Γ, the quotient $\Gamma\backslash\bar{\Omega}$ is isomorphic to the rigidification of a smooth projective geometrically connected curve over K_∞.*

PROOF: Assume first that Γ is one of the Γ_ν that arise in the isomorphism (15) for some admissible $\mathcal{K}$. The functor $X \mapsto X^{\mathrm{rig}}$ maps any smooth connected curve over K_∞ to a smooth connected rigid curve. Hence $\mathfrak{M}_{\mathcal{K},K_\infty}$ consists of exactly $h_{\mathcal{K}} := \operatorname{card}\mathrm{Cl}_{\mathcal{K}}$ connected components, which must all be projective. Thus each $\Gamma_\nu\backslash\bar{\Omega}$ arises from such a component via the functor $X \mapsto X^{\mathrm{rig}}$.

Let Γ be any arithmetic subgroup, and choose an A-lattice Λ of rank 2 in K and a non-zero ideal $\mathfrak{n}$ such that $\mathrm{Aut}_A(\Lambda,\mathfrak{n}) \subset \Gamma \subset \mathrm{Aut}_A(\Lambda)$. Clearly $\mathrm{Aut}_A(\Lambda,\mathfrak{n})$ arises as a Γ_ν for $\mathcal{K} = \mathcal{K}(\mathfrak{n})$, and so we let $X_{\Gamma,\mathfrak{n}}$ be any component of $\mathfrak{M}_{\mathcal{K}(\mathfrak{n}),K_\infty}$ such that $X^{\mathrm{rig}}_{\Gamma,\mathfrak{n}} \cong \Gamma\backslash\bar{\Omega}$ under the isomorphism (15).

The quotient $G := \Gamma/\mathrm{Aut}_A(\Lambda,\mathfrak{n})$ acts on $\mathrm{Aut}_A(\Lambda,\mathfrak{n})\backslash\bar{\Omega}$ in the obvious way. The corresponding action on $\mathfrak{M}_{\mathcal{K}(\mathfrak{n}),K_\infty}$ is given by an action on level structures, which in general is no longer free. For example via the correspondence given in the above theorem, one can see that this action fixes the component $X_{\Gamma,\mathfrak{n}}$. It follows easily that $\Gamma\backslash\bar{\Omega}$ is isomorphic to $(G\backslash X_{\Gamma,\mathfrak{n}})^{\mathrm{rig}}$. ∎

5 Drinfeld modular forms

This section is devoted to the definition of Drinfeld modular forms. Our treatment is based on [15], VII, [7] and [55]. The first two parts give a rapid introduction to modular forms on Ω and their re-interpretation in terms of harmonic cocycles by Teitelbaum. This is followed by a thorough discussion of modular forms on moduli spaces $\mathfrak{M}_{\mathcal{K}}$ where we use adelic language. Unfortunately, this is somewhat technical. However, the various reformulations of modular forms we give in this context will all be needed when we introduce and compare Hecke operators in various different settings.

Most results are well-known, but not available in the literature in the form needed. In [55], a characterization of double cusp forms in terms of harmonic cocycles was given. We rework this in detail and obtain a similar characterization in terms of the Steinberg module. Furthermore, we define integral Steinberg cycles, which will be needed for the formulation of the Eichler-Shimura isomorphism in Section 10.

5.1 Modular forms for arithmetic subgroups

Let Γ be an arithmetic subgroup of $\mathrm{GL}_2(K)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$, $n, l \in \mathbb{Z}$ and $f\colon \Omega(\mathbb{C}_\infty) \to \mathbb{C}_\infty$, we define

$$(f \underset{n,l}{||} \gamma)(z) := f(\gamma z)(\det \gamma)^l (cz + d)^{-n}.$$

Note that $(f \,||_{n,l}\, \gamma) \,||_{n,l}\, \gamma' = f \,||_{n,l} (\gamma\gamma')$.

Definition 5.1 *A rigid analytic function $f\colon \Omega(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ is called a* modular function of weight n and type l (for Γ)*, if*

$$f \underset{n,l}{||} \gamma = f \quad \forall \gamma \in \Gamma. \tag{16}$$

We write $F_{n,l}(\Gamma)$ for the space of such.

For $\gamma \in \mathrm{GL}_2(K)$ define the rational end $[\underline{s}_\gamma] = \gamma(0:1)$ of $\mathcal{T}$ and define c_γ as the corresponding cusp in $\Gamma\backslash\mathbb{P}^1(K)$. Let $\Gamma'_{\underline{s}} \subset \Gamma_{\underline{s}}$ be the maximal p'-torsion free subgroup. Then $\Gamma_\gamma := \gamma^{-1}\Gamma'_{\underline{s}}\gamma = \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in I_\gamma\right\}$ for some fractional almost-ideal I_γ of A. Define $\Omega_\gamma := \bigcup\{\mathfrak{U}_t : \Gamma_t \cap \Gamma_\gamma \text{ is non-trivial}\}$. If f is a modular function for Γ, then the definition shows that $f \,||_{n,l}\, \gamma^{-1}$ is invariant under Γ_γ, and in particular, the restriction of $f \,||_{n,l}\, \gamma$ to Ω_γ induces a rigid analytic function $f_\gamma\colon \Gamma_\gamma\backslash\Omega_\gamma(\mathbb{C}_\infty) \to \mathbb{C}_\infty$.

By Lemma 3.31, $\Gamma_\gamma\backslash\Omega_\gamma$ is an affinoid subdomain of $\mathbb{P}^1$ and by Theorem 3.25 there is a neighborhood of the image of the cusp c_γ which is a punctured disc. Therefore the function f_γ has a Laurent-series expansion - possibly with infinite principal part. It is easy to see that the index of the lowest non-vanishing term of the Laurent series expansion of f_γ only depends on the cusp c_γ.

In analogy with the classical situation, the following notions are introduced:

Definition 5.2 *An element $f \in F_{n,l}(\Gamma)$ is a* modular form*, if for all $\gamma \in \mathrm{GL}_2(K)$, the function f_γ has vanishing principal part.*

A modular form is called a cusp form*, respectively* double cusp form*, if for all $\gamma \in \mathrm{GL}_2(K)$, the function f_γ vanishes at the cusp c_γ to the order at least* 1*, respectively at least* 2*.*

The $\mathbb{C}_\infty$-vector space of modular forms of weight n and type l for Γ is denoted by $M_{n,l}(\Gamma)$, the corresponding space of cusp forms by $S_{n,l}(\Gamma)$, and that of double cusp forms by $S^2_{n,l}(\Gamma)$.

General convention: In the case $l = n-1$, we drop the subscript l. This convention will also be applied to the doubly indexed spaces defined later.

Remark 5.3 The Laurent series expansion of f_γ can be made explicit by using the reciprocal of the function $e_{I_\gamma}(z) := \prod_{i\in I_\gamma}(1-z/i)$ as a local coordinate near the image of c_γ on $\Gamma_\gamma\backslash\Omega_\gamma$. Namely, for z 'sufficiently close' to $[\underline{s}]$, the function $f\,||_{n,l}\,\gamma$ has an expansion $\sum a_n e_{I_\gamma}(z)^{-n}$.

Note that while a priori, the weight can be any integer, the type should be thought of as an element in $\mathbb{Z}/(l_\Gamma)$, where l_Γ is the order of the subgroup $\det(\Gamma)$ of k^*. If $l_\Gamma = 1$ it follows that for fixed n, all $M_{n,l}$ are isomorphic, and the same holds for the $S_{n,l}$ and the $S^2_{n,l}$.

Proposition 5.4 ([55], Lem. 15) *Let $g(\Gamma)$ denote the genus of $\Gamma\backslash\bar{\Omega}$ and $h(\Gamma)$ the number of cusps of $\Gamma\backslash\Omega$. If Γ is p'-torsion free, then*

$$\dim_{\mathbb{C}_\infty} S_n(\Gamma) = (n-1)(g(\Gamma)+h(\Gamma)-1).$$

By [55], Lemma 15, and the simple observation that any arithmetic subgroup Γ contains a p'-torsion free arithmetic subgroup Γ', one obtains the following.

Corollary 5.5 *For no arithmetic $\Gamma \subset \mathrm{GL}_2(K)$, there exist cusp forms of weight less than* 2.

As in the case of classical modular forms, one can interpret modular forms as sections of suitable line bundles on $\Gamma\backslash\Omega$. We only indicate this in the case when Γ is p'-torsion free. Define an action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ on $\mathcal{O}_\Omega$ as follows. A section f on an affinoid U is mapped to $\frac{1}{cz+d} f\circ\gamma$ which is a section on $\gamma^{-1}U$.

To see that $\omega_\Gamma := \Gamma\backslash\mathcal{O}_\Omega$ exists as a line bundle on $\Gamma\backslash\Omega$, it suffices to consider the action of Γ_t on $\mathcal{O}_{\mathfrak{U}_t}$ for any simplex $t \in T$. If Γ_t is trivial, there is nothing to show. Otherwise, we may assume that t is a simplex of the unstable component $\mathcal{T}_{\underline{s}}$ of the rational end $[\underline{s}] = (0:1)$, i.e., that $\Gamma_t \subset \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in K\right\} \cap \Gamma$. The transformation property of a modular function shows that Γ_t acts trivially on $\omega_\Gamma\Big|_{\mathfrak{U}_t}$. Hence ω_Γ exists.

As we assume that Γ is p'-torsion free, it is not hard to show that ω_Γ has an extension $\bar{\omega}_\Gamma$ to $\Gamma\backslash\bar{\Omega}$, which near the cusp c_γ is isomorphic to $\mathcal{O}_{\Gamma_\gamma\backslash\bar{\Omega}_\gamma}$. Let [cusps] denote the divisor $\sum[c]$ where the sum is over all cusps of $\Gamma\backslash\bar{\Omega}$. With the above definitions in place, we obtain the following:

Proposition 5.6 *For any $n \in \mathbb{N}_0$, there are canonical isomorphisms*

$$\begin{aligned} M_n(\Gamma) &\cong H^0(\Gamma\backslash\bar{\Omega}, \bar{\omega}_\Gamma^{\otimes n}), \\ S_n(\Gamma) &\cong H^0(\Gamma\backslash\bar{\Omega}, \bar{\omega}_\Gamma^{\otimes n}([\mathrm{cusps}])), \\ S^2_n(\Gamma) &\cong H^0(\Gamma\backslash\bar{\Omega}, \bar{\omega}_\Gamma^{\otimes n}(2[\mathrm{cusps}])). \end{aligned}$$

5.2 Harmonic cocycles

Definition 5.7 *Let M be an abelian group. An M-valued function $c\colon \mathcal{T}_1^o \to M$ on the oriented edges of $\mathcal{T}$ is called a* harmonic cocycle *if* *(i) for all vertices v of $\mathcal{T}$,*

$$\sum_{e\in\mathcal{T}_1^o, t(e)=v} c(e) = 0.$$

(ii) for all oriented edges $e \in \mathcal{T}_1^o$ on has $c(-e) = -c(e)$.

We now define various representations of an arithmetic group Γ and an A-lattice $\Lambda \subset K^2$ of rank 2 such that $\Gamma \subset \mathrm{Aut}_A(\Lambda)$. Recall that K^2 is a space of row vectors and that $\gamma \in \mathrm{GL}_2(K)$ acts on $v \in K^2$ by $v \mapsto \gamma \circ v := v\gamma^{-1}$. We denote by Ω_A the module of differentials of A, and view it as a trivial $\mathrm{GL}_2(K)$-module. For a projective (not necessarily finitely generated) A-module P, we define its dual $P^* := \mathrm{Hom}_A(P, A)$. If P carries a right action of $\mathrm{GL}_2(K)$, then so does P^* in a natural way.

Let now R be an A-algebra and $n \geq 2$. We define the representation $V_{n,l}(\Lambda \otimes_A R)$ of Γ as

$$V_{n,l}(\Lambda \otimes_A R) := \Big((\det \Lambda)^{l+1-n} \otimes_A \mathrm{Sym}^{n-2}\,\mathrm{Hom}_A(\Lambda, \Omega_A)\Big)^* \otimes_A R.$$

This is a projective left R-module of rank $n-1$. We write $\gamma_{n,l}$ for the operation of an element $\gamma \in \Gamma$ on $V_{n,l}(\Lambda \otimes_A R)$. If $R = A$, we simply write $V_{n,l}(\Lambda)$. We fix an isomorphism $K \cong \Omega_K \cong \Omega_A \otimes_A K$. For $K \subset R$, this shows that $V_{n,l}(\Lambda \otimes_A R)$ is independent of Λ, and so we simply write $V_{n,l}(R)$. If $K_\infty \subset R$, then there is a natural action for any $\gamma \in \mathrm{GL}_2(K_\infty)$ on $V_{n,l}(R)$, which we also denote by $\gamma_{n,l}$.

Remark 5.8 We chose a different normalization than the one given in [55], but we also have a different action on Λ and below we also use a slightly different residue map. The reason will eventually become apparent, when we prove the Eichler-Shimura isomorphism, cf. Theorem 10.3.

Definition 5.9 *Let $\Gamma \subset \mathrm{GL}_2(K)$ be arithmetic and assume that R contains K. We define $C^{\mathrm{har}}_{n,l}(\Gamma, R)$ as the set of Γ-invariant harmonic cocycles which take values in $V_{n,l}(R)$, and call them R-*valued harmonic cocycles of weight n and type l (on Γ).

According to our general convention we write $C^{\mathrm{har}}_{n}(\Gamma, R)$ for $C^{\mathrm{har}}_{n,n-1}(\Gamma, R)$ and $V_n(\Lambda \otimes R)$ for and $V_{n,n-1}(\Lambda \otimes R)$.

In order to construct the residue map

$$\mathrm{Res}_\Gamma \colon S_{k,l}(\Gamma) \to C^{\mathrm{har}}_{k,l}(\Gamma, \mathbb{C}_\infty)$$

from [55], we first have to explain the notion of residue for a holomorphic function on $\Omega(\mathbb{C}_\infty)$ and an oriented edge $e := \overrightarrow{vv'}$ of $\mathcal{T}$. Let $\bar{e}$ be the edge $\{v, v'\}$ and choose an isomorphism between $\mathfrak{U}_{\bar{e}}$ and $\mathrm{Ann} := \{z \in \mathbb{C}_\infty : 1 \leq |z|_\infty \leq q^{1/3}\}$ such that the boundary $|z|_\infty = 1$ comes from the part of $\mathfrak{U}_{\bar{e}}$ that lies near v and the boundary $|z|_\infty = q^{1/3}$ from the part near v'. Such an isomorphism is not unique. However if we fix a differential ω on Ann and consider its (convergent) Laurent expansion $\omega = \sum_{n\in\mathbb{Z}} a_n z^n\,dz$, then the coefficient a_{-1} is unchanged under automorphisms of Ann which preserve the boundary components. In analogy with the classical terminology, a_{-1} is called the residue of ω. In particular, for any holomorphic function h on $\mathfrak{U}_{\bar{e}}$, the notion of a residue of $h\,dz$ on $\mathfrak{U}_{\bar{e}}$ relative to e is well-defined. We write $\mathrm{Res}_e\, h\,dz$.

Suppose $R = \mathbb{C}_\infty$ and let X, Y be the dual basis of the standard basis f_1, f_2 of K^2. Then $\mathrm{Sym}^{n-2}(\mathrm{Hom}(K^2, \mathbb{C}_\infty))$ is the set of homogeneous polynomials over $\mathbb{C}_\infty$ in X, Y of degree $n-2$. Hence an element of $V_{n,l}(\mathbb{C}_\infty)$ is determined by the values it takes on the basis $\mathsf{X}^i\mathsf{Y}^{n-2-i}$, $i=0,\ldots,n-1$. One easily checks that for $w\in(\mathrm{Sym}^{n-2}\operatorname{Hom}(K^2,\mathbb{C}_\infty))^*$ and $\gamma\in\mathrm{GL}_2(K_\infty)$ one has

$$(\gamma_{n,l}w)(\mathsf{X}^i\mathsf{Y}^{n-2-i})=\det\gamma^{1-l}w((d\mathsf{X}-c\mathsf{Y})^i(-b\mathsf{X}+a\mathsf{Y})^{n-2-i}).$$

For $f\in S_{n,l}(\Gamma)$ and an oriented edge e of $\mathcal{T}^o$, we define an element $(\operatorname{Res}f)(e)$ in $V_{n,l}(\mathbb{C}_\infty)$ by

$$(\operatorname{Res}f)(e)(\mathsf{X}^i\mathsf{Y}^{n-2-i}):=\operatorname{Res}_e(-z)^{n-2-i}f(z)\,dz.$$

As we have a different normalization as the one given in [55], let us compute the effect of the residue map on $f\,\big|\big|_{n,l}$. Let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2(K_\infty)$. Then

$$\begin{aligned}
&\gamma_{n,l}(\operatorname{Res}(f\underset{n,l}{||}\gamma))(e)(\mathsf{X}^i\mathsf{Y}^{n-2-i})\\
=\;&(\operatorname{Res}(f\underset{n,l}{||}\gamma))(e)(\det\gamma^{1-l}((d\mathsf{X}-c\mathsf{Y})^i(-b\mathsf{X}+a\mathsf{Y})^{n-2-i}))\\
=\;&\operatorname{Res}_e(f\underset{n,l}{||}\gamma)(z)\det\gamma^{1-l}(d+cz)^i(-b-az)^{n-2-i}\,dz\\
=\;&\operatorname{Res}_e f(\gamma z)\det\gamma^l(cz+d)^{-n}\det\gamma^{1-l}(d+cz)^i(-b-az)^{n-2-i}\,dz\\
=\;&\operatorname{Res}_{\gamma e}f(w)(-w)^{n-2-i}\,dw\\
=\;&(\operatorname{Res}f)(\gamma e)(X^iY^{n-2-i})
\end{aligned}$$

Thus we find

$$\operatorname{Res}(f\underset{n,l}{||}\gamma)(e)=\gamma_{n,l}^{-1}(\operatorname{Res}f)(\gamma e).\tag{17}$$

This shows that $\operatorname{Res}f$ takes its values in $C^{\mathrm{har}}_{n,l}(\Gamma,\mathbb{C}_\infty)$. But more is true:

Theorem 5.10 ([55], Thm. 16) *The map* $\operatorname{Res}_\Gamma: S_{n,l}(\Gamma)\to C^{\mathrm{har}}_{n,l}(\Gamma,\mathbb{C}_\infty)$ *is an isomorphism.*

5.3 The Steinberg module

Let Γ be p'-torsion free. There is an alternative description of harmonic cocycles which makes essential use of a complex based on the stable simplices of $\mathcal{T}$. Define $\mathcal{T}_0^{\mathrm{st}},\mathcal{T}_1^{\mathrm{st}},\mathcal{T}_1^{o,\mathrm{st}}$ as the set of stable vertices, edges and oriented edges of $\mathcal{T}$ with respect to Γ, respectively. For a vertex v of $\mathcal{T}$, we define $[v]\in\mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$ to be zero, if v is unstable, and to be the symbol corresponding to v, if v is stable. For $e=\overrightarrow{vv'}\in\mathcal{T}_1^{o,\mathrm{st}}$, we define

$$\partial_\Gamma[e]:=[v']-[v].$$

Note that the vertices adjacent to a stable edge are not necessarily stable themselves. However, if a vertex is stable, then so are all adjacent edges.

In slight abuse of notation, we define $\mathbb{Z}[\overline{\mathcal{T}}_1^{o,\mathrm{st}}]:=\mathbb{Z}[\mathcal{T}_1^{o,\mathrm{st}}]/\langle[-e]+[e]:e\in\mathcal{T}_1^{o,\mathrm{st}}\rangle$. We continue to use the notation $[e]$ for symbols in $\mathbb{Z}[\overline{\mathcal{T}}_1^{o,\mathrm{st}}]$. Because $\partial_\Gamma[-e]=-\partial_\Gamma[e]$, there is an induced map from $\mathbb{Z}[\overline{\mathcal{T}}_1^{o,\mathrm{st}}]$ to $\mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$. The *stable complex for* Γ is defined as

$$\mathcal{C}^{\mathrm{st}}_{\Gamma,\bullet}:\quad\ldots\longleftarrow 0\longleftarrow\mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]\xleftarrow{\partial_\Gamma}\mathbb{Z}[\overline{\mathcal{T}}_1^{o,\mathrm{st}}]\longleftarrow 0\longleftarrow\ldots\tag{18}$$

By the very definition of stability and the finiteness of the number of Γ-orbits of stable simplices, the $\mathbb{Z}[\Gamma]$-modules $\mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$ and $\mathbb{Z}[\overline{\mathcal{T}}_1^{o,\mathrm{st}}]\cong\mathbb{Z}[\mathcal{T}_1^{\mathrm{st}}]$ are free of finite rank. In [48], II.Thm. 13, it is shown that ∂_Γ is surjective. Hence the Steinberg module St_Γ of Γ defined by

$$0 \longrightarrow \mathrm{St}_\Gamma \longrightarrow \mathbb{Z}[\overline{\mathcal{T}}_1^{o,\mathrm{st}}] \xrightarrow{\partial_\Gamma} \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}] \longrightarrow 0 \tag{19}$$

is a finitely generated, projective $\mathbb{Z}[\Gamma]$-module.

For any group G, let us abbreviate $\otimes_{\mathbb{Z}[G]}$ by $\otimes_G$. Note that for a left G-module M and a right G-module N, the module $N \otimes_G M$ is isomorphic to the covariants $(N \otimes M)_G$.

For $n \geq 2$ and any K-algebra R, there is a map $\Phi_\Gamma \colon C_n^{\mathrm{har}}(\Gamma, R) \to \mathbb{Z}[\overline{\mathcal{T}}_1^{o,\mathrm{st}}] \otimes_\Gamma V_n(R)$ defined by

$$\Phi_\Gamma(c) := \sum_{e \in \Gamma \backslash \mathcal{T}_1^{o,\mathrm{st}}/\{\pm 1\}} [e] \otimes c(e).$$

The indexing set of the sum is formed by choosing one oriented edge in each class of $\Gamma \backslash \mathcal{T}_1^{\mathrm{st}}$. This map is well-defined because c is Γ-invariant. The harmonicity of c implies that Φ_Γ takes its image in $\mathrm{St}_\Gamma \otimes_\Gamma V_n(\Lambda \otimes R)$. The following is shown in [55], Lem. 20:

Proposition 5.11 *If Γ is p'-torsion free and R is a K-algebra, the map*

$$\Phi_\Gamma \colon C_n^{\mathrm{har}}(\Gamma, R) \to \mathrm{St}_\Gamma \otimes_\Gamma V_n(R) \cong H_1(\mathcal{C}_{\Gamma,\bullet}^{\mathrm{st}} \otimes_\Gamma V_n(R))$$

is an isomorphism.

That Φ_Γ is an isomorphisms implies in particular, that a harmonic cocycle is determined by its values on the stable edges. Therefore it must be possible to express the value on any edge in terms of the values of suitable stable edges. The following taken from [55], p. 506, explains how to do this:

Definition 5.12 *For an edge $e \in \mathcal{T}_1^o$, define its source, which is a subset of $\mathcal{T}_1^o$, as follows: If e is stable, then $\mathrm{src}(e) = \{e\}$. If e is unstable, then $\mathrm{src}(e)$ is the set of all $e' \in \mathcal{T}_1^{o,\mathrm{st}}$ such that*

(i) there exists an unstable vertex v' of e' such that e lies on the half line from v' to $b(v')$, where b is defined below Def. 3.22, and

(ii) e' has the same orientation as e.

Lemma 5.13 *For $c \in C_{n,l}^{\mathrm{har}}(R)$ and $e \in \mathcal{T}_1^o$ one has $c(e) = \sum_{e' \in \mathrm{src}(e)} c(e')$.*

Based on Proposition 5.11, the following generalizes Definition 5.9 for groups Γ which are p'-torsion free.

Definition 5.14 *Let Γ be p'-torsion free, $\Lambda \subset K^2$ a lattice such that $\Gamma \subset \mathrm{Aut}_A(\Lambda)$, and R an A-ring. We define the module of R-*valued Steinberg cycles of weight n (on Γ, relative to Λ,) *as*

$$\mathbf{C}_n^{\mathrm{St}}(\Gamma, \Lambda \otimes R) := \mathrm{St}_\Gamma \otimes_\Gamma V_n(\Lambda \otimes R).$$

As St_Γ is a finitely generated projective $\mathbb{Z}[\Gamma]$-module, it follows easily that $\mathbf{C}_n^{\mathrm{St}}(\Gamma, \Lambda \otimes_A R)$ is a finitely generated projective R-module.

As a last result on the Steinberg-module, we describe a complex which is quasi-isomorphic to $\mathcal{C}_{\Gamma,\bullet}^{\mathrm{st}}$, which will be needed in the application in later sections.

For an oriented edge e, define the corresponding non-oriented edge as $\bar{e}$ and the map

$$\tilde{\partial}_\Gamma \colon \mathbb{Z}[\mathcal{T}_1^{o,\mathrm{st}}] \longrightarrow \mathbb{Z}[\mathcal{T}_1^{\mathrm{st}}] \oplus \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}] : [e] \mapsto ([\bar{e}], [t(e)]).$$

Recall that we have $[v] = 0$ if v is an unstable vertex. Define for any Γ the complex

$$\tilde{\mathcal{C}}^{\mathrm{st}}_{\Gamma,\bullet}: \quad \ldots \longleftarrow 0 \longleftarrow \mathbb{Z}[\mathcal{T}_1^{\mathrm{st}}] \oplus \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}] \xleftarrow{\tilde{\partial}_\Gamma} \mathbb{Z}[\mathcal{T}_1^{o,\mathrm{st}}] \longleftarrow 0 \longleftarrow \ldots \tag{20}$$

Proposition 5.15 *The diagram*

$$\begin{tikzcd}
\mathbb{Z}[\overline{\mathcal{T}}_1^{\mathrm{st}}] \arrow[r, "\partial_\Gamma"] \arrow[d, "\alpha_1:[e]\mapsto[e]-[-e]"'] & \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}] \arrow[d, "\alpha_0:[v]\mapsto(0,[v])"] \\
\mathbb{Z}[\mathcal{T}_1^{o,\mathrm{st}}] \arrow[r, "\tilde{\partial}_\Gamma"] & \mathbb{Z}[\mathcal{T}_1^{\mathrm{st}}] \oplus \mathbb{Z}[\mathbb{Z}_0^{\mathrm{st}}]
\end{tikzcd}$$

defines a quasi-isomorphism $\mathcal{C}^{\mathrm{st}}_{\Gamma,\bullet} \longrightarrow \tilde{\mathcal{C}}^{\mathrm{st}}_{\Gamma,\bullet}$.

Proof: For the diagram to define a morphism of complexes, we have to show that it commutes. This follows from

$$\begin{aligned}
\tilde{\partial}_\Gamma \alpha_1([e]) &= \tilde{\partial}_\Gamma([e] - [-e]) = (0, [t(e)]) - (0, [t(-e)]) \\
&= \alpha_0([t(e)] - [t(-e)]) = \alpha_0 \partial_\Gamma([e]).
\end{aligned}$$

The morphism of complexes $\alpha_\bullet$ is injective on objects. Therefore to see that it defines a quasi-isomorphism, we need to show that the cokernel is an exact complex. But the cokernel is easily identified with

$$\mathbb{Z}[\mathcal{T}_1^{\mathrm{st}}] \xrightarrow{\mathrm{id}} \mathbb{Z}[\mathcal{T}_1^{\mathrm{st}}],$$

where the map on the 1-chains sends $[e]$ to $[\bar{e}]$ and the map on 0-chains maps a pair $([\bar{e}], [v])$ to the symbol $[\bar{e}]$. ∎

5.4 Double cusp forms and Steinberg cycles

We will give a characterization of double cusp forms in terms of Steinberg cycles. Again we assume throughout that Γ is p'-torsion free.

We first recall the definition of St_Γ as given in [48], II.2, for an arithmetic subgroup Γ of $\mathrm{GL}_2(K)$. Define $\deg_\Gamma : \mathbb{Z}[\mathbb{P}^1(K)] \longrightarrow \mathbb{Z} : \sum n_{\underline{s}}[\underline{s}] \mapsto \sum n_{\underline{s}}$. This map is clearly Γ-equivariant, if we let Γ act on $\mathbb{Z}[\mathbb{P}^1(K)]$ via its usual action on ends, and on $\mathbb{Z}$ via the trivial action.

Proposition 5.16 ([48], § 2.9) *As a* $\mathbb{Z}[\Gamma]$*-module* St_Γ *is isomorphic to the kernel of* $\deg_\Gamma$*.*

Proof: As in Section 3, the subgraph of $\mathcal{T}$ formed by the unstable simplices will be denoted $\mathcal{T}_\infty$. The symbols $\{v\}$ denote the elements of the canonical basis of $\mathbb{Z}[\mathcal{T}_0]$. They are not to be confused with the symbols $[v]$ used in the definition of ∂_Γ. Again we define the boundary map

$$\partial\colon \mathcal{T}_1^o \to \mathbb{Z}[\mathcal{T}_0] : e = \overrightarrow{vv'} \mapsto \{v'\} - \{v\},$$

which extends linearly to $\mathbb{Z}[\mathcal{T}_1^o]$ and whose restriction to $\mathbb{Z}[\mathcal{T}^o_{\infty,1}]$ maps to $\mathbb{Z}[\mathcal{T}_{\infty,0}]$. As in the previous subsection, we define the modules $\mathbb{Z}[\overline{\mathcal{T}}^o_0]$ and $\mathbb{Z}[\overline{\mathcal{T}}^o_{\infty,1}]$ as the quotients of the respective modules without a bar by the submodules generated by the chains $[e] + [-e]$. Clearly the map ∂ induces a map on these, too.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}[\overline{\mathcal{T}}^o_{\infty,1}] \arrow[r] \arrow[d, "\partial"] & \mathbb{Z}[\overline{\mathcal{T}}^o_1] \arrow[r] \arrow[d, "\partial"] & \mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_1] \arrow[r] \arrow[d, "\partial_\Gamma"] & 0 \\
0 \arrow[r] & \mathbb{Z}[\mathcal{T}_{\infty,0}] \arrow[r] & \mathbb{Z}[\mathcal{T}_0] \arrow[r, "\beta"] & \mathbb{Z}[\mathcal{T}^{\mathrm{st}}_0] \arrow[r] & 0.
\end{tikzcd}$$

It is commutative and compatible with the action of Γ. Since $\mathcal{T}$ is a (connected) tree, the middle arrow is injective with cokernel $\mathbb{Z}$. The map to $\mathbb{Z}$ is given by mapping $\sum n_v\{v\}$ to $\sum n_v$.

Because $\mathcal{T}_\infty$ is a collection of trees, the left hand vertical map is injective, too. Recall that the map b defined a labeling of the components of $\mathcal{T}_\infty$. To describe the cokernel, we define the Γ-equivariant map

$$\tilde{b}\colon \mathbb{Z}[\mathcal{T}_{\infty,0}] \to \mathbb{Z}[\mathbb{P}^1(K)] : \sum_{v\in\mathcal{T}_{\infty,0}} n_v\{v\} \mapsto \sum n_v[b(v)]. \tag{21}$$

By the definition of b, the map $\tilde{b}$ is surjective with kernel $(\mathbb{Z}[\overline{\mathcal{T}}^o_{\infty,1}], \partial)$.

Regarding the right hand vertical map, we have seen earlier that it is surjective and its kernel was defined to be St_Γ. Now the Snake Lemma yields the short exact sequence

$$0 \longrightarrow \mathrm{St}_\Gamma \xrightarrow{\alpha_\Gamma} \mathbb{Z}[\mathbb{P}^1(K)] \xrightarrow{\deg_\Gamma} \mathbb{Z} \longrightarrow 0 \tag{22}$$

of $\mathbb{Z}[\Gamma]$-modules, where α_Γ denotes the connecting morphism. This proves the proposition. ∎

To make the map α_Γ explicit, fix an element $\sum n_e[e] \in \mathrm{St}_\Gamma$. Via $\mathcal{T}_1^{o,\mathrm{st}} \hookrightarrow \mathcal{T}_1^o$, this can be regarded as an element in $\mathbb{Z}[\overline{\mathcal{T}}^o_1]$. Being in the kernel of ∂_Γ means that $\partial(\sum n_e[e]) \in \mathrm{Ker}(\beta)$, where β is as in the above diagram. In other words, all symbols $\{v\}$, $v \in \mathcal{T}_0$, that appear in $\partial(\sum n_e[e]) \in \mathrm{Ker}(\beta)$ with a non-zero coefficient, belong to a vertex in $\mathcal{T}_\infty$. To such vertices, we apply the map b to obtain an end, i.e., an element of $\mathbb{P}^1(K)$.

This shows that α_Γ can be described as the restriction of the map $\mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_1] \to \mathbb{Z}[\mathbb{P}^1(K)]$ to St_Γ, which on symbols $[e]$, $e \in \mathcal{T}_1^{o,\mathrm{st}}$, is defined as follows: First the symbol $[e]$ is mapped to the sum over the symbols of its unstable boundary vertices with the appropriate signs. (The sum consists of at most two terms.) Then this linear combination is mapped via the map $\tilde{b}$ defined in (21) to a linear combination of symbols of the corresponding ends.

Definition 5.17 *Let Γ be p'-torsion free, $\Lambda \subset K^2$ a lattice such that $\Gamma \subset \mathrm{Aut}_A(\Lambda)$, and R and A-algebra. We define $\mathbf{C}^{\mathrm{St},2}_n(\Gamma, \Lambda\otimes R)$ as the kernel of*

$$\alpha_\Gamma \otimes \mathrm{id}\colon \mathrm{St}_\Gamma \otimes_\Gamma V_n(\Lambda\otimes R) \xrightarrow{\alpha_\Gamma} \mathbb{Z}[\mathbb{P}^1(K)] \otimes_\Gamma V_n(\Lambda\otimes R),$$

and call it the module of doubly cuspidal Steinberg cycles over R of weight n and level Γ (relative to Λ).

For $R = K$, and thus for any K-algebra R, the module $\mathbf{C}^{\mathrm{St},2}_n(\Gamma, \Lambda\otimes R)$ is projective over R. However for general A, this will not be the case.

Proposition 5.18 *Suppose that Γ is p'-torsion free and R is a field containing K. Then*

$$\dim_R \mathbf{C}^{\mathrm{St},2}_n(\Gamma, R) = \begin{cases} g(\Gamma) & \text{if } n = 2 \\ (n-2)(g(\Gamma)+h(\Gamma)-1)+g(\Gamma)-1 & \text{if } n > 2 \end{cases}$$

PROOF: If we tensor the short exact sequenc (22) over $\mathbb{Z}[\Gamma]$ with $V_n(R)$ and use the above definition, we obtain the following four term exact sequence:

$$0 \to \mathbf{C}_n^{\mathrm{St},2}(\Gamma, R) \to \mathbf{C}_n^{\mathrm{St}}(\Gamma, R) \to \mathbb{Z}[\mathbb{P}^1(K)] \otimes_\Gamma V_n(R) \to \mathbb{Z} \otimes_\Gamma V_n(R) \to 0$$

The last term vanishes unless $n = 2$ in which case it is isomorphic to R. If $\{\Gamma_c : c \in \text{cusps}\}$ is a set of representatives of stabilizer subgroups for the cusps of $\Gamma\backslash\Omega$, then

$$\mathbb{Z}[\mathbb{P}^1(K)] \otimes_\Gamma V_n(R) \cong \bigoplus_c V_n(R)_{\Gamma_c}.$$

One can show that each of the terms $V_n(R)_{\Gamma_c}$ is isomorphic to R, cf. Remark 5.23. The formula is now easily derived from Proposition 5.4. ∎

The following theorem justifies the above definition:

Theorem 5.19 *The isomorphism $S_n(\Gamma) \cong \mathbf{C}_n^{\mathrm{St}}(\Gamma, \mathbb{C}_\infty)$ given by $\Phi_\Gamma \circ \mathrm{Res}_\Gamma$ restricts to an isomorphism $S_n^2(\Gamma) \cong \mathbf{C}_n^{\mathrm{St},2}(\Gamma, \mathbb{C}_\infty)$.*

The above theorem and Proposition 5.18 yield:

Corollary 5.20 *Suppose that Γ is p'-torsion free. Then*

$$\dim_{\mathbb{C}_\infty} S_n^2(\Gamma, \mathbb{C}_\infty) = \begin{cases} g(\Gamma) & \text{if } n = 2 \\ (n-2)(g(\Gamma) + h(\Gamma) - 1) + g(\Gamma) - 1 & \text{if } n > 2 \end{cases}$$

For the proof of Theorem 5.19, we will need the following well-known lemma, whose simple proof is omitted.

Lemma 5.21 *Let G be some group and W a $K[G]$-module, which is finite dimensional over K. Then $(W^*)_G \cong (W^G)^*$ via the map which arises from $W^* \to (W^G)^* : f \mapsto f_{|W^G}$.*

Recall that $V_n(\mathbb{C}_\infty)$ is defined as the dual of the set $\mathbb{C}_\infty[\mathsf{X}, \mathsf{Y}]_{n-2}$ of homogeneous polynomials in $\mathbb{C}_\infty[\mathsf{X}, \mathsf{Y}]$ of degree $n-2$ with the (left) action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ given by

$$\gamma(\mathsf{X}) = a\mathsf{X} + c\mathsf{Y} \quad \gamma(\mathsf{Y}) = b\mathsf{X} + d\mathsf{Y}.$$

Let $[\underline{s}] = (0 : 1)$ with $\Gamma_{\underline{s}} = \left\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) : a \in I_{\underline{s}}\right\}$ for some fractional almost-ideal $I_{\underline{s}}$ of A.

Corollary 5.22 *The vector space $\mathbb{C}_\infty[\mathsf{X}, \mathsf{Y}]_{n-2}^{\Gamma_{\underline{s}}}$ has X^{n-2} as a basis. We identify its dual with $\mathbb{C}_\infty$ by evaluation at X^{n-2}. Then the isomorphism*

$$V_n(\mathbb{C}_\infty)_{\Gamma_{\underline{s}}} \xrightarrow{\cong} (\mathbb{C}_\infty[\mathsf{X}, \mathsf{Y}]_{n-2}^{\Gamma_{\underline{s}}})^* \cong \mathbb{C}_\infty$$

is given by evaluating a functional $v \in V_n(\mathbb{C})$ at X^{n-2}.

PROOF: The second assertion is immediate from the first and the previous lemma. So it remains to identify the space $\mathbb{C}_\infty[\mathsf{X}, \mathsf{Y}]_{n-2}^{\Gamma_{\underline{s}}}$, which obviously contains $\mathbb{C}_\infty \mathsf{X}^{n-2}$.

Let P be a homogeneous polynomial of degree $n-2$ which is invariant under $\Gamma_{\underline{s}}$ and consider $p(y) := p(1, y)$. Then $p(y) = p(y - a)$ for all $a \in I_{\underline{s}}$. Since $I_{\underline{s}}$ is infinite, p must be constant, and so P must lie in $\mathbb{C}_\infty \mathsf{X}^{n-2}$. ∎

Remark 5.23 The argument used in the proof of the previous corollary also shows the following: Suppose F is an A-field (possibly finite) and that under the map $\mathrm{Aut}(\Lambda) \mapsto \mathrm{Aut}(\Lambda \otimes_A F)$, the image of the p-group $\Gamma_{\underline{s}}$ contains at least $n-1$ elements. Then $V_n(\Lambda \otimes_A F)_{\Gamma_{\underline{s}}}$ is one-dimensional.

Lemma 5.24 *For $i = 1, \ldots, l$, let ζ_i be in $\mathbb{C}_\infty$ and r_i in $\mathbb{R}_{>0}$, such that the 'closed' discs $D_i := \{z \in \mathbb{C}_\infty : |z - \zeta_i|_\infty \leq r_i\}$ are disjoint. By $\overset{\circ}{D}_i$ we denote the 'open' disc $\{z \in \mathbb{C}_\infty : |z - \zeta_i|_\infty < r_i\}$. Let f be holomorphic on $D := \mathbb{P}^1 - \bigcup \overset{\circ}{D}_i$ with $f(\infty) = 0$. Then*

$$f'(\infty) = \sum_i \mathrm{Res}_{\partial D_i} f(z)\, dz,$$

where the $\partial D_i = D_i \smallsetminus \overset{\circ}{D}_i$ are oriented toward ∞ (see proof) and f is given near ∞ in local coordinates $1/z$.

Proof: Choose radii $r_i' > r_i$ such that the corresponding discs $D_i' \supset D_i$ are still disjoint. Represent f on the annulus $A_i := \{z \in \mathbb{C}_\infty : |z - \zeta_i|_\infty \in [r_i, r_i']\}$ as a convergent Laurent series

$$\sum_{n \in \mathbb{Z}} \tilde{a}_{n,i}(z - \zeta_i)^n.$$

Then $\mathrm{Res}_{\partial D_i} f(z)\mathrm{d}z = \tilde{a}_{-1,i}$.

By [12], I.1.3 Prop., on D the function f can be uniquely written as a convergent series

$$f = \sum_{n=1}^{\infty} \sum_{i=1}^{l} a_{-n,i}(z - \zeta_i)^{-n}. \tag{23}$$

On the annulus A_i, the functions $(z - \zeta_j)^{-n}$, $j \neq i$, are holomorphic. Comparing principal parts, it follows that $\tilde{a}_{n,i} = a_{n,i}$ for $n < 0$. Differentiating (23) and evaluating at ∞ yields:

$$-\lim_{z \to \infty} f'(z) = \sum_{i=1}^{n} a_{-1,i} = \sum_i \mathrm{Res}_{\partial D_i} f(z)\mathrm{d}z.$$

However in local coordinates $w = \frac{1}{z}$ at infinity, one has $\sum b_n w^n = \sum b_n z^{-n}$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}w} f_{|w=0} = -\lim_{z \to \infty} f'(z),$$

and the asserted equality follows. ∎

Let f be in $S_n(\Gamma)$. Fix $[\underline{s}] = (0:1)$, $\Gamma_{\underline{s}}$ and $I_{\underline{s}}$ as above Corollary 5.22. Let $\mathcal{T}_{\underline{s}}$ be the subtree of simplices t with $b(t) = (0:1)$, and define $\Omega_{\underline{s}} \subset \Omega$ as the union of all affinoid subdomains $\mathfrak{U}_t$ where t is a simplex of $\mathcal{T}_{\underline{s}}$. On these two objects, there is a natural action of $\Gamma_{\underline{s}}$, and we had seen that f descends to a holomorphic function $f_{\underline{s}}$ on $D_{\underline{s}} := \Gamma_{\underline{s}} \backslash \Omega_{\underline{s}}$. The space $D_{\underline{s}}$ was identified as an affinoid subdomain of $\mathbb{P}^1$ with ∞ removed in Lemma 3.31. The point ∞ corresponds to the cusp of $[\underline{s}]$. Because f is cuspidal, it extends by zero holomorphically to this cusp.

For any end $[\underline{s}]$, we define the source of $[\underline{s}]$ as

$$\begin{aligned} \mathrm{src}([\underline{s}]) &:= \bigcup \{\mathrm{src}(e) : e \in \mathcal{T}^o_{\underline{s},1},\ e \text{ points toward } [\underline{s}]\} \\ &= \{e \in \mathcal{T}_1^{o,\mathrm{st}} : b(t(e)) = [\underline{s}]\}. \end{aligned}$$

Lemma 5.25 *Let the notation be as above.*

(a) The group $\Gamma_{\underline{s}}$ acts freely on $\mathrm{src}([\underline{s}])$ *with finitely many orbits, say represented by edges $e_1, \ldots, e_l$.*

(b) For $[\underline{s}] = (0:1)$, the map

$$\mathbf{C}_n^{\mathrm{har}}(\Gamma, \mathbb{C}_\infty) \stackrel{\cong}{\to} \mathbf{C}_n^{\mathrm{St}}(\Gamma, \mathbb{C}_\infty) \stackrel{\alpha_\Gamma \otimes \mathrm{id}}{\to} \bigoplus V_n(\mathbb{C}_\infty)_{\Gamma_i} \stackrel{\mathrm{pr}}{\to} V_n(\mathbb{C}_\infty)_{\Gamma_{\underline{s}}} \cong \mathbb{C}_\infty, \qquad (24)$$

where pr *is the projection onto the factor at $[\underline{s}]$, is given by sending a cocycle c to $\sum_{i=1}^{l} c(e_i)(\mathsf{X}^{n-2})$.*

(c) Suppose $c = \mathrm{Res}_\Gamma f$. Then $\sum_{i=1}^{l} c(e_i)(\mathsf{X}^{n-2}) = f'_{\underline{s}}(\infty)$, where at the cusp ∞ we take the local coordinate $w = 1/z$.

Proof: To prove (a), note first that $\mathcal{T}_{\underline{s}}$ is acted on by $\Gamma_{\underline{s}}$. Therefore the source of $[\underline{s}]$ admits an action of $\Gamma_{\underline{s}}$. Because the source consists of stable edges, this action must be free. To prove the finiteness assertion, we claim that the canonical map $\Gamma_{\underline{s}} \backslash \mathrm{src}([\underline{s}]) \to \Gamma \backslash \mathcal{T}_1^{\mathrm{st}}$ is injective. Since the set on the right is finite, this will finish the proof of (a).

To show the claim let e, e' both be in the source of $[\underline{s}]$ and assume there is a $\gamma \in \Gamma$ such that $\gamma e = e'$. Let $\underline{s}$ and $\underline{s}'$ be the half lines starting at e, respectively e' that both represent $[\underline{s}]$. Elements of Γ preserve stable and unstable simplices. Therefore $\gamma \underline{s}$ is a rational half line whose initial edge e' is stable and all of whose other edges are unstable. Furthermore e' points in the direction of the end $[\gamma \underline{s}]$. But then $\gamma \underline{s}$ is, except for e' the half line from the unstable vertex $t(e')$ to $b(t(e'))$. Because $\underline{s}'$ is another such half line, we must have $\gamma \underline{s} = \underline{s}'$, i.e., $\gamma \in \Gamma_{\underline{s}}$, as claimed.

For (b), recall that the first isomorphism in (24) sends c to

$$\sum_{e \in \Gamma \backslash \mathcal{T}_1^{o,\mathrm{st}} / \{\pm 1\}} [e] \otimes c(e).$$

Let us assume that the set of representatives $\Gamma \backslash \mathcal{T}_1^{o,\mathrm{st}} / \{\pm 1\}$ contains the edges e_i. For the map

$$\mathbf{C}_n^{\mathrm{St}}(\Gamma, \mathbb{C}_\infty) \to V_n(\mathbb{C}_\infty)_{\Gamma_{\underline{s}}}$$

these are the only edges that are relevant.

For each e_i denote by v_i its unique vertex that belongs to $\mathcal{T}_{\underline{s}}$. (Because $\mathcal{T}$ is a tree, and hence contains no loops, it is not possible that both ends of an edge belong to the same unstable region.) From our explicit description of α_Γ given above Definition 5.17, we see that the image of c in $V_n(\mathbb{C}_\infty)_{\Gamma_{\underline{s}}}$ is given as $\sum_i \{[\underline{s}]\} \otimes_{\Gamma_{\underline{s}}} c(e_i)$, where $\{[\underline{s}]\}$ is the symbol in $\mathbb{Z}[\mathbb{P}^1(K)]$ for the rational end $[\underline{s}]$. Lemma 5.22 completes the proof of (b)

For the last part, we recall that $(\mathrm{Res}_\Gamma f(z)\, dz)(e_i)$ is given as the residue of $f(z)\, dz$ on the annulus $\mathfrak{U}_{e_i}$ with the orientation inherited from e_i. Furthermore, this residue only depends on a neighborhood of $\mathfrak{U}_{e_i} \cap \mathfrak{U}_{v_i}$ inside $\mathfrak{U}_{v_i}$. Since e_i is stable, such a neighborhood can be chosen so that it maps biholomorphically into $D_{\underline{s}}$. Let $\partial_i D_{\underline{s}}$ denote the corresponding boundary part of $D_{\underline{s}}$ oriented toward ∞. Then we have

$$\mathrm{Res}_{\partial_i D_{\underline{s}}} f_\gamma(z)\, dz = (\mathrm{Res}_\Gamma f)(e_i).$$

The proof of (c) now follows from (b) and Lemma 5.24. ∎

Proof of Theorem 5.19: Via a suitable $\gamma \in \mathrm{GL}_2(K)$, we can move any end to $(0:1)$. This was precisely the way in which we defined cusp forms and double cusp forms in Definition 5.2. Therefore the above lemma applies to all cusps and it shows that f is a double cusp form of weight n for Γ if and only if its image lies in the kernel of $\alpha_\Gamma \otimes \mathrm{id}_{V_n(\mathbb{C}_\infty)}$, which is precisely $\mathbf{C}_n^{\mathrm{St},2}(\Gamma, \mathbb{C}_\infty)$. ∎

Finally we want to comment on the integrality properties of $\mathbf{C}_n^{\mathrm{St},2}(\Gamma, \Lambda \otimes R)$.

Proposition 5.26 *Fix $n \in \mathbb{N}$. Suppose there exists a non-zero ideal $\mathfrak{n}$ of A such that*

(i) $\Gamma \supset \Gamma(\mathfrak{n})$,

(ii) the class group of $A(\mathfrak{n})$ is trivial,

(iii) R is an $A(\mathfrak{n})$-algebra, and

(iv) for all $\mathfrak{p} \in \mathrm{Max}(A(\mathfrak{n}))$ the residue field $k_\mathfrak{p}$ has order at least $n-2$.

Then the module $\mathbf{C}_n^{\mathrm{St},2}(\Gamma, \Lambda \otimes R)$ is projective and of finite rank over R.

The same conclusion holds if either R is a Dedekind domain or flat over A.

Proof: The second assertion is rather obvious in the case that R is a Dedekind domain, since from the definition we have $\mathbf{C}_n^{\mathrm{St},2}(\Gamma, \Lambda \otimes R) \subset \mathbf{C}_n^{\mathrm{St}}(\Gamma, \Lambda \otimes R)$. The module on the right is projective and hence torsion free, and therefore the submodule $\mathbf{C}_n^{\mathrm{St},2}(\Gamma, \Lambda \otimes R)$ has the same properties.

Next we prove the first assertion. For this let $[\underline{s}_1], \ldots, [\underline{s}_r]$ be rational ends which give a complete set of representatives of the cusps, so that

$$\mathbb{Z}[\mathbb{P}^1(K)] \cong \bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_i}^{\Gamma} \mathbb{Z},$$

where we abbreviate $\Gamma_i := \Gamma_{[\underline{s}_i]}$.

We first claim that under our hypothesis, the modules $M_n := V_n(\Lambda \otimes A(\mathfrak{n}))_\Gamma$ and $M_{n,i} := V_n(\Lambda \otimes A(\mathfrak{n}))_{\Gamma_i}$ are projective over $A(\mathfrak{n})$. By Remark 5.23, we have $\mathrm{rank}\, M_{n,i} = 1$ for all i, n, $\mathrm{rank}\, M_2 = 1$ and $\mathrm{rank}\, M_n = 0$ for $n > 2$. Because $A(\mathfrak{n})$ is a Dedekind domain, it suffices to show that the ranks at the special fibers above all $\mathfrak{p} \not| \mathfrak{n}$ are the same as the generic rank.

Let us first consider the modules $M_n \otimes k_\mathfrak{p}$ for such a prime $\mathfrak{p}$, where the case $n = 2$ is trivial, and so we assume $n > 2$. By (i), we have $M_n \otimes k_\mathfrak{p} \subset ((\mathrm{Sym}^{n-2} k_\mathfrak{p}^2)^{\mathrm{SL}_2(k_\mathfrak{p})})^* = 0$, because $n - 2 < |k_\mathfrak{p}|$. r For the modules $M_{n,i} \otimes k_\mathfrak{p}$, we proceed as follows: By (ii), any projective module over $A(\mathfrak{n})$ is free. This implies that any Borel in $\mathrm{GL}_2(A(\mathfrak{n}))$ is conjugate (over this group) to a standard Borel. Hence it follows that $\Gamma_i = \Gamma \cap g B_0 g^{-1}$ for some $g \in \mathrm{GL}_2(A(\mathfrak{n}))$ and the standard Borel B_0 of $\mathrm{GL}_2(A(\mathfrak{n}))$. Because Γ is p'-torsion free, this and (i) imply that Γ_i modulo $\Gamma_i \cap \Gamma(\mathfrak{p})$ is isomorphic to $\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in k_\mathfrak{p}\}$. By Remark 5.23 and condition (iv), we have $\dim_{k_\mathfrak{p}} M_{n,i} \otimes k_\mathfrak{p} = 1$. This finishes the proof of the claim.

The claim implies that all the modules in the 4-term exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{C}_n^{\mathrm{St},2}(\Gamma, \Lambda \otimes A(\mathfrak{n})) &\longrightarrow \mathbf{C}_n^{\mathrm{St}}(\Gamma, \Lambda \otimes A(\mathfrak{n})) \\ &\longrightarrow \oplus_i V_n(\Lambda \otimes A(\mathfrak{n}))_{\Gamma_i} \longrightarrow V_n(\Lambda \otimes A(\mathfrak{n}))_\Gamma \longrightarrow 0 \end{aligned}$$

are projective. Therefore tensoring it with R over $A(\mathfrak{n})$, which is possible by (iii), yields the analogous 4-term sequence for R. But then

$$\mathbf{C}_n^{\mathrm{St},2}(\Gamma, \Lambda \otimes R) \cong \mathbf{C}_n^{\mathrm{St},2}(\Gamma, \Lambda \otimes A(\mathfrak{n})) \otimes_{A(\mathfrak{n})} R.$$

By the part already shown $\mathbf{C}_n^{\mathrm{St},2}(\Gamma, \Lambda\otimes A(\mathfrak{n}))$ is projective over $A(\mathfrak{n})$. The first assertion of the proposition is now obvious.

To prove the second assertion in the case where R is flat over A, one tensors again the above 4-term sequence with R over A. Because R is A-flat, the resulting sequence is exact. One concludes as in the previous paragraph. ∎

This concludes our discussion of Drinfeld modular forms in the local case. We now turn to the global situation.

5.5 Adelic modular forms

Our presentation borrows heavily from the discussion of Hilbert modular forms as given in [52]. We will develop the theory in global (i.e. adelic) as well as a local form. Fix an open subgroup $\mathcal{K}$ of $\mathrm{GL}_2(\hat{A})$ and define x_ν, Γ_ν as in (14). We will frequently make use of the isomorphism

$$\xi\colon \coprod_\nu \Gamma_\nu\backslash\Omega \xleftarrow{\cong} \mathrm{GL}_2(K)\backslash(\Omega\times\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}),$$

where the right hand side can be identified with a coarse moduli space for $\mathcal{M}_{\mathcal{K}}^{\mathrm{rig}}$.

Regarding the notation we make the following conventions. Elements in $\mathrm{GL}_2(\mathbb{A})$ will be denoted by $w=(w_f,w_\infty)$, where $w_f\in\mathrm{GL}_2(\mathbb{A}^f)$ and $w_\infty\in\mathrm{GL}_2(K_\infty)$. We also use the symbol g for elements in $\mathrm{GL}_2(\mathbb{A}^f)$, g_{f} for an element in $\mathcal{K}$ and g_∞ for elements in $\mathrm{GL}_2(K_\infty)$. Finally elements in $\mathrm{GL}_2(K)$ are usually denoted by α or γ.

Definition 5.27 *(Local Definition) The space of* global modular functions *of level $\mathcal{K}$, weight n and type l is defined as*

$$\underline{F}_{n,l}(\mathcal{K}) := \prod_{\nu\in\mathrm{Cl}_{\mathcal{K}}} F_{n,l}(\Gamma_\nu).$$

The respective subspaces of modular forms, cusp forms, *and* double cusp forms *are defined as*

$$\underline{M}_{n,l}(\mathcal{K}) := \prod_\nu M_{n,l}(\Gamma_\nu),\quad \underline{S}_{n,l}(\mathcal{K}) := \prod_\nu S_{n,l}(\Gamma_\nu),\quad \underline{S}^2_{n,l}(\mathcal{K}) := \prod_\nu S^2_{n,l}(\Gamma_\nu).$$

A modular form is denoted by $\underline{f}=(f_\nu)=(f_\nu)_{\nu\in\mathrm{Cl}_{\mathcal{K}}}\in\underline{M}_{n,l}(\mathcal{K})$. Furthermore, we define $l_{\mathcal{K}}$ as the least common multiple of the orders of the groups $\det\Gamma_\nu$, $\nu\in\mathrm{Cl}_{\mathcal{K}}$, so that $\underline{M}_{n,l}(\mathcal{K})=\underline{M}_{n,l'}(\mathcal{K})$ whenever $l\equiv l' \pmod{l_{\mathcal{K}}}$.

To prepare the global definition of modular forms, we need yet another description of $(\Omega\times\mathrm{GL}_2(\mathbb{A}^f))/\mathcal{K}$. Let $(g_{\mathrm{f}},g_\infty)\in\mathcal{K}\times\mathrm{GL}_2(K_\infty)$ act from the right and $\alpha\in\mathrm{GL}_2(K)$ from the left on $\Omega\times\mathrm{GL}_2(\mathbb{A})$ by

$$(z,w_{\mathrm{f}},w_\infty)\mapsto(g_\infty^{-1}z,\alpha w_{\mathrm{f}}g_{\mathrm{f}},\alpha w_\infty g_\infty).$$

The map $(z,w_{\mathrm{f}},w_\infty)\mapsto(w_\infty z,w_{\mathrm{f}})$ induces a $\mathrm{GL}_2(K)$-equivariant bijection

$$(\Omega\times\mathrm{GL}_2(\mathbb{A}))/(\mathcal{K}\times\mathrm{GL}_2(K_\infty))\longrightarrow\Omega\times\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}.$$

We extend the x_ν to elements of $\mathrm{GL}_2(\mathbb{A})$ by defining their component at ∞ as the identity. Thus the x_ν also form a set of representatives of $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})/(\mathcal{K}\times\mathrm{GL}_2(K_\infty))$. Finally, we set $\mathfrak{X}_{\mathcal{K}}:=\Omega(\mathbb{C}_\infty)\times\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})/\mathcal{K}$.

To $\underline{f}$ we associate a $\mathbb{C}_\infty$-valued function $\mathbf{f}$ on $\Omega(\mathbb{C}_\infty) \times \mathrm{GL}_2(\mathbb{A})$ by

$$\mathbf{f}(z, \alpha x_\nu w) := \Big(f_\nu \underset{n,l}{||} w_\infty\Big)(z)$$

for $z \in \Omega(\mathbb{C}_\infty)$, $\alpha \in \mathrm{GL}_2(K)$ and $w = (w_{\mathrm{f}}, w_\infty)$. To see that this map is well-defined, one needs the invariance property of f_ν under Γ_ν from the definition of a modular form. As $\mathrm{GL}_2(K)$ does not act on Ω but only on $\mathrm{GL}_2(\mathbb{A})$ via the action defined above, the function $\mathbf{f}$ is a map from $\Omega(\mathbb{C}_\infty) \times \mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})/\mathcal{K}$ to $\mathbb{C}_\infty$. Clearly one can recover $\underline{f}$ from $\mathbf{f}$ via

$$f_\nu(z) := \mathbf{f}(z, x_\nu). \tag{25}$$

For $w \in \mathrm{GL}_2(\mathbb{A})$, the function $z \mapsto \mathbf{f}(z, w)$ is a holomorphic function from $\Omega(\mathbb{C}_\infty)$ to $\mathbb{C}_\infty$. For $g_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$, we define $\mathbf{f} \,||_{n,l}\, g_\infty$ by

$$\Big(\mathbf{f} \underset{n,l}{||} g_\infty\Big)(z, w) := \mathbf{f}(g_\infty z, w)(\det g_\infty)^l (cz + d)^{-n},$$

and the right translation r_{g_∞} by

$$r_{g_\infty}\mathbf{f}(z, w_{\mathrm{f}}, w_\infty) = \mathbf{f}(z, w_{\mathrm{f}}, w_\infty g_\infty).$$

Definition 5.28 *(Global Definition) An* adelic modular function of level $\mathcal{K}$, weight n and type l *is a map* $\mathbf{f} : \mathfrak{X}_\mathcal{K} \to \mathbb{C}_\infty$ *which is holomorphic in the first variable and satisfies*

$$\mathbf{f} \underset{n,l}{||} g_\infty = r_{g_\infty}\mathbf{f} \quad \textit{for any } g_\infty \in \mathrm{GL}_2(K_\infty).$$

The above discussion shows that for fixed $\mathcal{K}, n, l$ there is a bijection between adelic modular functions and modular functions for $\mathcal{K}$. The $\mathbb{C}_\infty$-vector space of adelic modular forms is denoted $\mathbf{F}_{n,l}(\mathcal{K})$. One can use this correspondence to define adelic modular forms, adelic cusp forms, etc., as follows: For $w \in \mathrm{GL}_2(\mathbb{A}^f)$ let I_w be the fractional almost-ideal of A defined by

$$\Big\{b \in K : \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K) \cap w\mathcal{K}w^{-1}\Big\}.$$

Definition 5.29 *An adelic modular function* $\mathbf{f}$ *is an* adelic modular form, adelic cusp form, adelic double cusp form *if for all* $w \in \mathrm{GL}_2(\mathbb{A}^f)$ *the Laurent expansion*

$$\mathbf{f}(z, w, 1) = \sum_{n \in \mathbb{Z}} a_n e_{I_w}(z)^{-n}, \tag{26}$$

which converges near the cusp $(0 : 1)$*, satisfies* $a_n = 0$ *for* $n < 0$*,* $n < 1$*,* $n < 2$*, respectively.*

The corresponding $\mathbb{C}_\infty$*-vector spaces are denoted* $\mathbf{M}_{n,l}(\mathcal{K})$*,* $\mathbf{S}_{n,l}(\mathcal{K})$*,* $\mathbf{S}^2_{n,l}(\mathcal{K})$*, respectively.*

The main reason for introducing the global view point is that it facilitates the discussion of Hecke operators to be given in the following section, and that it is unlike the local definition free of any choices, like choosing the x_ν. We omit an interpretation of modular forms for $\mathcal{M}_\mathcal{K}$ as sections of certain line bundles, as it will not be needed in the sequel.

The following is immediate from the above discussion:

Proposition 5.30 *There are natural isomorphisms*

$$\mathbf{M}_{n,l}(\mathcal{K}) \cong \underline{M}_{n,l}(\mathcal{K}) \quad \mathbf{S}_{n,l}(\mathcal{K}) \cong \underline{S}_{n,l}(\mathcal{K}) \quad \mathbf{S}^2_{n,l}(\mathcal{K}) \cong \underline{S}^2_{n,l}(\mathcal{K}).$$

For an admissible subgroup $\mathcal{K}$, let $g(\mathcal{K})$ denote the sum over the genus of the components of $\overline{\mathfrak{M}}_{\mathcal{K},K^{\mathrm{alg}}}$. Let furthermore $h(\mathcal{K})$ denote the number of cusps of $\overline{\mathfrak{M}}_{\mathcal{K},K^{\mathrm{alg}}}$, i.e., $\mathrm{card}(\overline{\mathfrak{M}}_{\mathcal{K},K^{\mathrm{alg}}} \smallsetminus \mathfrak{M}_{\mathcal{K},K^{\mathrm{alg}}})$. Also, let $d(\mathcal{K})$ be the number of connected components of $\overline{\mathfrak{M}}_{\mathcal{K},K^{\mathrm{alg}}}$. Finally let $s_n(\mathcal{K}) := \dim \underline{S}_{n,l}(\mathcal{K})$ and $s_n^2(\mathcal{K}) := \dim \underline{S}^2_{n,l}(\mathcal{K})$. The previous proposition combined with Proposition 5.4 and Corollary 5.20 implies:

Corollary 5.31 *If $\mathcal{K}$ is admissible, then $s_n(\mathcal{K}) = (n-1)(g(\mathcal{K}) + h(\mathcal{K}) - d(\mathcal{K}))$ and*

$$s_n^2(\mathcal{K}) = \begin{cases} g(\mathcal{K}) & \text{if } n = 2 \\ (n-2)(g(\mathcal{K}) + h(\mathcal{K}) - d(\mathcal{K})) + g(\mathcal{K}) - d(\mathcal{K}) & \text{if } n > 2 \end{cases}$$

While our local definition of modular form implies that the set of such is equal if the type is changed by adding a multiple of $l_{\mathcal{K}}$, this is no longer the case for the global description.

Lemma 5.32 *Suppose $l, l' \in \mathbb{Z}$ are congruent modulo $l_{\mathcal{K}}$. Then the assignment $\mathbf{f} \mapsto \mathbf{g}$ given by*

$$\mathbf{g}(z, \alpha x_\nu w) := \mathbf{f}(z, \alpha x_\nu w)(\det w_\infty)^{l'-l},$$

where $\alpha \in \mathrm{GL}_2(K)$, $w = (w_{\mathrm{f}}, w_\infty) \in \mathcal{K} \times \mathrm{GL}_2(K_\infty)$ and $\nu \in \mathrm{Cl}(\mathcal{K})$, defines for fixed $\mathcal{K}, n$ an isomorphism from the space of modular functions of type l' to that of type l. It restricts to isomorphisms

$$\mathbf{M}_{n,l}(\mathcal{K}) \cong \mathbf{M}_{n,l'}(\mathcal{K}), \quad \mathbf{S}_{n,l}(\mathcal{K}) \cong \mathbf{S}_{n,l'}(\mathcal{K}), \quad \mathbf{S}^2_{n,l}(\mathcal{K}) \cong \mathbf{S}^2_{n,l'}(\mathcal{K}).$$

Remark 5.33 Choosing a sign-function, i.e., a homomorphism $\mathrm{sign}: K_\infty^* \to k_\infty^*$ which is the identity on k^*, one could define $(f\,||_{n,l}\,\gamma)(z) := f(\gamma z)(cz+d)^n \,\mathrm{sign}(\det\gamma)^l$. Then the sets of global modular forms of types l, l' would automatically agree whenever $l \equiv l' \pmod{q_\infty - 1}$. So at the expense of this choice and various other somewhat less natural definitions, one could reduce the amount of redundancy described in the previous lemma. As there seems to be no definition that avoids this redundancy completely we opted for a simple definition with lots of redundancy.

In the number theoretic case, one can avoid all this as taking the square root and squaring are natural inverses for the positive real numbers. Over K_∞ taking square roots is no longer such a natural operation. To avoid its definition, types are introduced.

5.6 Adelic harmonic cocycles

As in the local situation for $\Gamma\backslash\Omega$, we now turn to a reformulation in terms of harmonic cocycles. The group $\mathrm{GL}_2(K)$ acts on $\mathcal{T} \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ on the left by

$$\alpha(t, g\mathcal{K}) = (\alpha t, \alpha g\mathcal{K}) \quad \alpha \in \mathrm{GL}_2(K).$$

Furthermore on $\mathcal{T} \times \mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})/\mathcal{K}$ there is a right action by $\mathrm{GL}_2(K_\infty)$ defined as

$$(t, \mathrm{GL}_2(K)g\mathcal{K})g_\infty = (g_\infty^{-1}t, \mathrm{GL}_2(K)gg_\infty\mathcal{K}) \quad \text{for } g \in \mathrm{GL}_2(\mathbb{A}^f), g_\infty \in \mathrm{GL}_2(K_\infty).$$

The following identifications will be important

$$\coprod_{\nu\in\mathrm{Cl}_{\mathcal{K}}} \Gamma_\nu\backslash\mathcal{T} \quad \stackrel{\cong}{\longleftarrow} \quad \mathrm{GL}_2(K)\backslash(\mathcal{T} \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}) \tag{27}$$

$$\stackrel{\cong}{\longleftarrow} \quad \Big(\mathcal{T} \times \mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})/\mathcal{K}\Big)/\mathrm{GL}_2(K_\infty), \tag{28}$$

where the first map sends a tuple $(t, \alpha x_\nu g_{\mathrm{f}})$ to $(\alpha^{-1}t)$ on the summand ν and the second a tuple $(t, w_{\mathrm{f}}, w_\infty)$ to $(w_\infty t, w_{\mathrm{f}})$, where t is a simplex of $\mathcal{T}$, $w_\infty \in \mathrm{GL}_2(K_\infty)$, $w_{\mathrm{f}} \in \mathrm{GL}_2(\mathbb{A}^f)$ and $\alpha \in \mathrm{GL}_2(K)$.

Definition 5.34 *Let R be a K_∞-algebra. An* adelic harmonic cocycle $\mathbf{c}$ of weight n, type l and level $\mathcal{K}$ over R *is a* $\mathrm{GL}_2(K_\infty)$*-invariant element* $\mathbf{c}$ *in*

$$\mathrm{Maps}(\mathcal{T}_1^o \times \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}, V_{n,l}(\mathbb{C}_\infty))$$

such that

(i) for all vertices v of $\mathcal{T}$, and all $w \in \mathrm{GL}_2(\mathbb{A})$

$$\sum_{e \in \mathcal{T}_1^o, t(e)=v} \mathbf{c}(e, w) = 0,$$

(ii) for all oriented edges $e \in \mathcal{T}_1^o$ on has $\mathbf{c}(-e, w) = -\mathbf{c}(e, w)$.

The R-module of all such adelic cocycles is denoted $\mathbf{C}_{n,l}^{\mathrm{har}}(\mathcal{K}, R)$.

Note that we let $\mathrm{GL}_2(K_\infty)$ act on $\mathrm{Maps}(\mathcal{T}_1^o \times \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}, V_{n,l}(\mathbb{C}_\infty))$ on the right, i.e., an element $g_\infty \in \mathrm{GL}_2(K_\infty)$ acts as

$$(\mathbf{c}g_\infty)(e, w) = (g_\infty)_{n,l}^{-1} \mathbf{c}(g_\infty e, w g_\infty^{-1}).$$

Using the isomorphisms (28) and (27), one can easily show the following.

Lemma 5.35 *Let R be a K_∞-algebra. For each tuple c_ν, $\nu \in \mathrm{Cl}_{\mathcal{K}}$, of harmonic cocycles of weight n and type l for Γ_ν define*

$$\mathbf{c}(e, \alpha x_\nu w) := (w_\infty^{-1})_{n,l} c_\nu(w_\infty e)$$

for $(e, \alpha x_\nu w) \in \mathcal{T}_1^o \times \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}$ and $w \in \mathcal{K} \times \mathrm{GL}_2(\infty)$. Then this is a bijection

$$\prod_\nu C_{n,l}^{\mathrm{har}}(\Gamma_\nu, R) \cong \mathbf{C}_{n,l}^{\mathrm{har}}(\mathcal{K}, R).$$

In particular $\mathbf{C}_{n,l}^{\mathrm{har}}(\mathcal{K}, R)$ is finitely generated projective over R.

As in the local situation, we define a residue map

$$\mathrm{Res}\, \mathbf{f} \colon \mathcal{T}_1^o \times \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K} \to V_{n,l}(\mathbb{C}_\infty)$$

which to a cusp form $\mathbf{f}$ of level $\mathcal{K}$ weight n and type l assigns

$$(\mathrm{Res}\, \mathbf{f})(e, w)(\mathsf{X}^i \mathsf{Y}^{n-2-i}) := \mathrm{Res}_e (-z)^{n-i-2} \mathbf{f}(z, w)\, dz \text{ for } e \in \mathcal{T}_1^o, w \in \mathrm{GL}_2(\mathbb{A}).$$

Lemma 5.36 *The following diagram commutes*

$$\begin{tikzcd}
(f_\nu)_{\nu \in \mathrm{Cl}_{\mathcal{K}}} \arrow[r, mapsto] \arrow[d, mapsto] & \mathbf{f} \arrow[d, mapsto] \\
(c_\nu := \mathrm{Res}\, f_\nu)_{\nu \in \mathrm{Cl}_{\mathcal{K}}} \arrow[r, mapsto] & \mathbf{c} = \mathrm{Res}\, \mathbf{f}.
\end{tikzcd}$$

We omit the proof which is easy if tedious. Regarding the assignment $\mathbf{f} \mapsto \mathrm{Res}\, \mathbf{f}$, we note that the condition $\mathbf{f} \,\big|\big|_{n,l}\, g_\infty = r_{g_\infty} \mathbf{f}$ for all $g_\infty \in \mathrm{GL}_2(K_\infty)$ translates into the $\mathrm{GL}_2(K_\infty)$-equivariance of the map $\mathrm{Res}\, \mathbf{f}$.

Based on Theorem 5.10, the following is a direct consequence of our definitions.

Theorem 5.37 *The assignment* $\mathbf{f} \mapsto \operatorname{Res} \mathbf{f}$ *defines an isomorphism* $\mathbf{S}_{n,l}(\mathcal{K}) \cong \mathbf{C}^{\mathrm{har}}_{n,l}(\mathcal{K}, \mathbb{C}_\infty)$.

Unlike for adelic Drinfeld modular forms, there is also a good description of adelic harmonic cocycles as maps on the finite adeles only, i.e. on $\mathcal{T}_1^o \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$. The benefit of this description is, that it can be used to define cusp forms over any ring which contains K.

Lemma 5.38 *For any* K_∞*-algebra* R*, there is a natural bijection*

$$\begin{aligned} &\mathrm{Maps}_{\mathrm{GL}_2(K_\infty)}(\mathcal{T}_1^o \times \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}, V_{n,l}(R)) \\ \cong\quad &\mathrm{Maps}_{\mathrm{GL}_2(K)}(\mathcal{T}_1^o \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}, V_{n,l}(R)), \end{aligned}$$

given by the following assignments. To an element $\mathbf{c}$ *on the left, we attach an element* $\tilde{\mathbf{c}}$ *on the right by*

$$\tilde{\mathbf{c}}(e, w_{\mathrm{f}}) := \mathbf{c}(e, w_{\mathrm{f}}, 1),$$

and to an element $\tilde{\mathbf{c}}$ *on the right, we attach an element* $\mathbf{c}$ *on the left by*

$$\mathbf{c}(e, w_{\mathrm{f}}, w_\infty) := (w_\infty)^{-1}_{n,l} \tilde{\mathbf{c}}(w_\infty e, w_{\mathrm{f}}).$$

Again we omit the details of the simple proof and only remark that an element $\alpha \in \mathrm{GL}_2(K)$ acts on a map $\tilde{\mathbf{c}}$ on the left via

$$(\alpha\tilde{\mathbf{c}})(e, w_{\mathrm{f}}) = \alpha_{n,l}(\tilde{\mathbf{c}}(\alpha^{-1}e, \alpha^{-1}w_{\mathrm{f}})).$$

Definition 5.39 *Let* R *be a* K*-algebra. An* adelic harmonic cocycle $\tilde{\mathbf{c}}$ of weight n, type l and level $\mathcal{K}$ over R *is an element* $\tilde{\mathbf{c}}$ *in*

$$\mathrm{Maps}_{\mathrm{GL}_2(K)}(\mathcal{T}_1^o \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}, V_{n,l}(R))$$

such that

(i) for all vertices v *of* $\mathcal{T}$ *and all* $g \in \mathrm{GL}_2(\mathbb{A}^f)$,

$$\sum_{e \in \mathcal{T}_1^o, t(e)=v} \tilde{\mathbf{c}}(e, g) = 0.$$

(ii) for all oriented edges $e \in \mathcal{T}_1^o$ *and all* $g \in \mathrm{GL}_2(\mathbb{A}^f)$, $\tilde{\mathbf{c}}(-e, g) = -\mathbf{c}(e, g)$.

We write $\tilde{\mathbf{C}}^{\mathrm{har}}_{n,l}(\mathcal{K}, R)$ *for the* R*-module of all such cocycles.*

Based on the previous lemma, it is simple to verify that this is a natural extension of Definition 5.34. This justifies that we call the elements of $\tilde{\mathbf{C}}^{\mathrm{har}}_{n,l}(\mathcal{K}, R)$ again adelic harmonic cocycles. The distinction will always be made in the notation by putting a tilde or not.

The following proposition compares the above module with the projective R-modules $C^{\mathrm{har}}_{n,l}(\Gamma_\nu, R)$. The simple proof is left to the reader.

Proposition 5.40 *For a tuple* c_ν, $\nu \in \mathrm{Cl}_\mathcal{K}$*, of harmonic cocycles of weight* n *and type* l *for* Γ_ν*, define*

$$\tilde{\mathbf{c}}(e, \alpha x_\nu g) := \alpha c_\nu(\alpha^{-1} e)$$

for $(e, \alpha x_\nu g) \in \mathcal{T}_1^o \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ *and* $g \in \mathcal{K}$*. For any* K*-algebra* R*, this yields a bijection*

$$\prod_\nu C^{\mathrm{har}}_{n,l}(\Gamma_\nu, R) \cong \tilde{\mathbf{C}}^{\mathrm{har}}_{n,l}(\mathcal{K}, R),$$

and so in particular $\tilde{\mathbf{C}}^{\mathrm{har}}_{n,l}(\mathcal{K}, R)$ *is finitely generated projective over* R.

As for adelic modular forms, one can easily prove the following.

Lemma 5.41 *Suppose $l, l' \in \mathbb{Z}$ are congruent modulo $l_{\mathcal{K}}$. Let R be a K-algebra. Then the assignment $\tilde{\mathbf{c}} \mapsto \tilde{\mathbf{c}}'$ given by*

$$\tilde{\mathbf{c}}'(e, \alpha x_\nu g) := \tilde{\mathbf{c}}(e, \alpha x_\nu g)(\det \alpha)^{l'-l}$$

defines for fixed $\mathcal{K}, n, R$ an isomorphism from $\mathbf{C}^{\mathrm{har}}_{n,l'}(\mathcal{K}, R)$ to $\mathbf{C}^{\mathrm{har}}_{n,l}(\mathcal{K}, R)$.

5.7 The adelic Steinberg module

We now assume that $\mathcal{K}$ is admissible. In particular, this implies that all Γ_ν are p'-torsion free, so that we can drop the index l. Our goal here is to define cusp and double cusp forms for $\mathcal{M}_{\mathcal{K}}$ with coefficients over any A-algebra. To do this, we will reformulate Definition 5.14 in adelic language, which needs various preparations. The main point is that we want to obtain definitions which are independent of various choices.

Define $\mathcal{T}_{\mathcal{K}} := \mathcal{T} \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ and $\mathcal{T}_\nu := \mathcal{T} \times \{x_\nu \mathcal{K}\} \cong \mathcal{T}$ for any $\nu \in \mathrm{Cl}_{\mathcal{K}}$. The i-simplices of $\mathcal{T}_{\mathcal{K}}$ are denoted by $\mathcal{T}_{\mathcal{K},i}$, the oriented edges by $\mathcal{T}^o_{\mathcal{K},1}$, and analogously for the $\mathcal{T}_\nu$.

Definition 5.42 *The stable i-simplices of $\mathcal{T}_{\mathcal{K}}$, and $\mathcal{T}_\nu$, respectively, are defined as*

$$\mathcal{T}^{\mathrm{st}}_{\mathcal{K},i} := \{(t, g\mathcal{K}) \in \mathcal{T}_{\mathcal{K},i} : \mathrm{Stab}_{\mathrm{GL}_2(K)}((t, g\mathcal{K})) \textit{ is trivial.}\}$$
$$\mathcal{T}^{\mathrm{st}}_{\nu,i} := \{t \in \mathcal{T}_{\nu,i} : \mathrm{Stab}_{\Gamma_\nu}(t) \textit{ is trivial.}\}$$

We often write $\tilde{t}$, $\tilde{e}$, $\tilde{v}$ for simplices, edges and vertices of $\mathcal{T}_{\mathcal{K}}$, respectively.

As $\mathcal{T}^{\mathrm{st}}_{\mathcal{K},i} = \bigoplus_\nu \mathrm{Ind}^{\mathrm{GL}_2(K)}_{\Gamma_\nu} \mathcal{T}^{\mathrm{st}}_{\nu,i}$ for $i = 0, 1$, the $\mathbb{Z}[\mathrm{GL}_2(K)]$-modules $\mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},i}]$ are free and finitely generated. We use the notation $\mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}]$ in the same way as in Subsection 5.3, so that this module is isomorphic to $\mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},1}]$. If we define the map $\partial_{\mathcal{K}} \colon \mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}] \to \mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},0}]$ in analogy to (18), then the surjectivity of the ∂_{Γ_ν} implies that of $\partial_{\mathcal{K}}$. Define $\mathrm{St}_{\mathcal{K}}$ by the short exact sequence

$$0 \longrightarrow \mathrm{St}_{\mathcal{K}} \longrightarrow \mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}] \xrightarrow{\partial_{\mathcal{K}}} \mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},0}] \longrightarrow 0. \tag{29}$$

This sequence splits as a sequence of $\mathbb{Z}[\mathrm{GL}_2(K)]$-modules as the object on the right is free and hence projective. Because $\mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}]$ is free and finitely generated over $\mathbb{Z}[\mathrm{GL}_2(K)]$, one has:

Proposition 5.43 *As a module over $\mathbb{Z}[\mathrm{GL}_2(K)]$, the group $\mathrm{St}_{\mathcal{K}}$ is finitely generated and projective. It is isomorphic to $\bigoplus_\nu \mathrm{Ind}^{\mathrm{GL}_2(K)}_{\Gamma_\nu} \mathrm{St}_{\Gamma_\nu}$.*

In the local situation, cf. Definition 5.14, we were able to define 'integral' harmonic cocycles, i.e., cocycles with values R for any A-algebra R. By the above proposition, for any K-algebra R there is an isomorphism

$$\mathrm{St}_{\mathcal{K}} \otimes_{\mathrm{GL}_2(K)} V_n(R) \cong \bigoplus_\nu \mathrm{St}_{\Gamma_\nu} \otimes_{\Gamma_\nu} V_n(R) \cong \bigoplus_\nu \mathbf{C}^{\mathrm{St}}_n(\Gamma_\nu, R).$$

This suggests the following definition in the global situation:

Definition 5.44 *Let R be a flat A-algebra and define $\Lambda_\nu := \hat{A}^2 x_\nu^{-1} \cap K^2$. A* Steinberg cycle Φ of weight n, and level $\mathcal{K}$ over R *is an element in the image of the injective map*

$$\bigoplus_\nu \mathbf{C}^{\mathrm{St}}_n(\Gamma_\nu, \Lambda_\nu \otimes_A R) \longrightarrow \mathrm{St}_{\mathcal{K}} \otimes_{\mathrm{GL}_2(K)} V_n(R \otimes_A K).$$

The module of all such cycles is denoted $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K},R)$.

The submodule spanned by the image of $\bigoplus_\nu \mathbf{C}_n^{\mathrm{St},2}(\Gamma_\nu,\Lambda_\nu\otimes_A R)$ *is denoted by* $\mathbf{C}_n^{\mathrm{St},2}(\mathcal{K},R)$ *and called the* R-module of doubly cuspidal Steinberg cycles Φ of weight n, and level $\mathcal{K}$ over R.

By their very definition, the modules $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K},R)$ and $\mathbf{C}_n^{\mathrm{St},2}(\mathcal{K},R)$ are finitely generated and projective over R. By flatness of R over A, the canonical map

$$\bigoplus_\nu \mathbf{C}_n^{\mathrm{St},2}(\Gamma_\nu,\Lambda_\nu\otimes_A R)\longrightarrow \mathbf{C}_n^{\mathrm{St},2}(\mathcal{K},R)$$

is always an isomorphism.

In Definition 5.44, we used special representatives of $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$, namely the elements x_ν. For these we defined Γ_ν-submodules Λ_ν inside K^2. Instead, we could take arbitrary coset representatives and define for $g\in\mathrm{GL}_2(\mathbb{A}^f)$ the group $\Gamma_g:=\mathrm{GL}_2(K)\cap g\mathcal{K}g^{-1}$ and the Γ_g-module $\Lambda_g:=\hat{A}^2g^{-1}\cap K^2$. The following proposition shows that the result is the same:

Proposition 5.45 *The above definitions are independent of any choice of representatives of* $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$*.*

At least for $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K},R)$, a short direct proof of the proposition can be given. However, we prefer to first introduce some more general machinery, which will also be useful later, and to then give the proof.

Definition 5.46 *A* local system $\bar{W}=(W,W_g)$ of left (right) modules for $\mathrm{GL}_2(\mathbb{A}^f)$ *consists of a left (right)* $\mathrm{GL}_2(K)$*-module* W *and* Γ_g*-submodules* W_g *for each* $g\in\mathrm{GL}_2(\mathbb{A}^f)$*, subject to the condition that for each* $g,g'\in\mathrm{GL}_2(\mathbb{A}^f)$ *with* $g'\in\alpha g\mathcal{K}$*,* $\alpha\in\mathrm{GL}_2(K)$*, one has*

$$\alpha(W_g)=W_{g'}\quad ((W_g)\alpha^{-1}=W_{g'}).$$

Lemma–Definition 5.47 *Let* $\bar{W}$ *be a local system of left modules for* $\mathrm{GL}_2(\mathbb{A}^f)$ *and let* $\bar{W}'$ *and* $\bar{Z}$ *be local systems of right modules for* $\mathrm{GL}_2(\mathbb{A}^f)$*. Assume that the natural map*

$$\bigoplus_{g\in\mathrm{Cl}_\mathcal{K}}\mathrm{Ind}_{\Gamma_g}^{\mathrm{GL}_2(K)}Z_g\to Z$$

is an isomorphism for any choice of a set of representatives of $\mathrm{Cl}_\mathcal{K}$*. Then the following hold:*

(a) The image of

$$\bigoplus_{g\in\mathrm{Cl}_\mathcal{K}}(Z_g\otimes_{\Gamma_g}W_g)\longrightarrow(Z\otimes_{\mathrm{GL}_2(K)}W),\quad i\geq 0,$$

denoted by $\bar{Z}\otimes_{\mathrm{GL}_2(K)}\bar{W}$*, is independent of the chosen representatives* g *of* $\mathrm{Cl}_\mathcal{K}$*. If the* Z_g *are projective over* $\mathbb{Z}[\Gamma_g]$*, then the above map is injective.*

(b) The map

$$\bigoplus_{g\in\mathrm{Cl}_\mathcal{K}}\mathrm{Hom}_{\Gamma_g}(Z_g,W'_g)\longrightarrow\mathrm{Hom}_{\mathrm{GL}_2(K)}(Z,W')\quad i\geq 0$$

is injective. The image is independent of of the chosen representatives g *of* $\mathrm{Cl}_\mathcal{K}$*, and is denoted by* $\mathrm{Hom}_{\mathrm{GL}_2(K)}(\bar{Z},\bar{W}')$*.*

(c) Suppose the W_g *are* $A[\Gamma_g]$*-modules which are finitely generated and projective over* A *and that* $W\cong W_g\otimes_A K$*. By* $\bar{W}^*$ *we denote the system of right modules defined by* $W_g^*:=\mathrm{Hom}(W_g,A)$*,* $W^*:=\mathrm{Hom}(W,K)$*, and where*

$\alpha \in \mathrm{GL}_2(K)$ *acts on* $w^* \in W^*$ *by* $(w^*\alpha)(w) = w^*(\alpha w)$. *Then the natural isomorphism*

$$\mathrm{Hom}_{\mathrm{GL}_2(K)}(Z, W^*) \cong (Z \otimes_{\mathrm{GL}_2(K)} W)^*$$

induces a perfect duality between the submodules

$$\mathrm{Hom}_{\mathrm{GL}_2(K)}(\bar{Z}, \bar{W}^*) \text{ and } \bar{Z} \otimes_{\mathrm{GL}_2(K)} \bar{W}.$$

We omit the simple proof and give instead some examples.

Example 5.48 Fix an integer $n \geq 2$ and an A-algebra R. In the applications, W', W'_g will always be the local system of left $\mathrm{GL}_2(\mathbb{A}^f)$-modules defined as

$$W'_g = \mathrm{Sym}^{n-2}(\mathrm{Hom}(\Lambda_g, \Omega_A) \otimes_A R), \quad W' := W'_g \otimes_A K.$$

For $\bar{W}$ we take system of right $\mathrm{GL}_2(\mathbb{A}^f)$-modules $(\bar{W}')^*$. Let us verify that the conditions in (c) are satisfied for $\bar{W}'$. By functoriality, it suffices to do this for $n = 3$.

Suppose that $g' \in \alpha g \mathcal{K}$ for some $\alpha \in \mathrm{GL}_2(K)$. Then

$$\Gamma_{g'} = \mathrm{GL}_2(K) \cap g'\mathcal{K}{g'}^{-1} = \mathrm{GL}_2(K) \cap \alpha g \mathcal{K} g^{-1}\alpha^{-1} = \alpha \Gamma_g \alpha^{-1}, \text{ and}$$

$$\Lambda_{g'} = \hat{A}^2 {g'}^{-1} \cap K^2 = \hat{A}^2 g^{-1}\alpha^{-1} \cap K^2 = (\hat{A}^2 g^{-1} \cap K^2)\alpha^{-1} = \alpha \circ \Lambda_g,$$

where we recall that we defined an action of $\mathrm{GL}_2(K)$ on the modules Λ_g below Proposition 4.10. Because $\alpha f = f \circ \alpha^{-1}$ for $f \in \mathrm{Hom}(\Lambda_g, \Omega_A)$, it follows easily that $W'_g \alpha^{-1} = W'_{g'}$.

It is now easy to check that $\bar{W} = (W, W_g)$ satisfies the conditions of the previous lemma, and for $R = A$ also the conditions of c).

Example 5.49 For Z, Z_g there will be various choices which satisfy the basic assumptions of the previous lemma, namely:

(i) $Z = \mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},i}]$, $Z_g = \mathbb{Z}[(\mathcal{T}_i \times \{g\mathcal{K}\})^{\mathrm{st}}]$, for $i = 0, 1$, with the obvious $\mathrm{GL}_2(K)$ and Γ_g actions. Note that in this case, the Z_g are free finitely generated $\mathbb{Z}[\Gamma_g]$-modules.

(ii) $Z = \mathbb{Z}[\mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}]$, $Z_g = \mathbb{Z}[(\mathcal{T}^o_1 \times \{g\mathcal{K}\})^{\mathrm{st}}]$.

(iii) $Z = \mathrm{St}_{\mathcal{K}}$, $Z_g = \mathrm{Ker}(\mathbb{Z}[(\overline{\mathcal{T}}^o_1 \times \{g\mathcal{K}\})^{\mathrm{st}}] \longrightarrow \mathbb{Z}[(\mathcal{T}_0 \times \{g\mathcal{K}\})^{\mathrm{st}}])$ again with the usual actions of $\mathrm{GL}_2(K)$ and Γ_g. The modules Z_g are finitely generated projective $\mathbb{Z}[\Gamma_g]$-modules.

(iv) $Z = \mathbb{Z}[\mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}]$ and $Z_g = \mathbb{Z}[\mathbb{P}^1(K) \times \{g\mathcal{K}\}]$, where the $\mathrm{GL}_2(K)$-action on $\mathbb{P}^1(K)$ is the action on ends, and the action on $\mathrm{GL}_2(\mathbb{A}^f)$ is the usual left action.

(v) $Z = \mathbb{Z}[\mathrm{Cl}_{\mathcal{K}}]$, and $Z_g = \mathbb{Z}[g]$, where g denotes the class of g. The action of $\mathrm{GL}_2(K)$ is trivial. Only the hypothesis for (a) hold.

(vi) In analogy to the complex in (20), one defines a complex which is quasi-isomorphic to the adelic Steinberg complex and globally takes the shape

$$\mathbb{Z}[\mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}] \longrightarrow \mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},0}] \oplus \mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},1}]$$

with the boundary map defined as in (20). Locally it is given by $\tilde{\mathcal{C}}^{\mathrm{st}}_{\Gamma_g,\bullet}$, where for given g, we identify $\mathcal{T}$ with $\mathcal{T} \times \{g\mathcal{K}\}$. The corresponding complex of local systems is denoted by $\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},\bullet}$.

PROOF of Proposition 5.45: The proof for $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, R)$ is obvious from the above lemma and the examples given. The proof for $\mathbf{C}_n^{\mathrm{St},2}(\mathcal{K}, R)$ follows in the same way by considering the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bigoplus_\nu \mathbf{C}_n^{\mathrm{St},2}(\Gamma_\nu, \Lambda_\nu \otimes_A R) \arrow[r] \arrow[d] & \mathbf{C}_n^{\mathrm{St},2}(\mathcal{K}, R \otimes K) \arrow[d] \\
\bigoplus_\nu \mathbf{C}_n^{\mathrm{St}}(\Gamma_\nu, \Lambda_\nu \otimes_A R) \arrow[r] \arrow[d] & \mathrm{St}_{\mathcal{K}} \otimes_{\mathrm{GL}_2(K)} V_n(R \otimes K) \arrow[d] \\
\bigoplus_\nu \mathbb{Z}[\mathbb{P}^1(K)] \otimes_{\Gamma_\nu} V_n(\Lambda_\nu \otimes R) \arrow[r] & \mathbb{Z}[\mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}] \otimes_{\mathrm{GL}_2(K)} V_n(R \otimes K),
\end{tikzcd}$$

where both columns are exact, and where $\mathrm{GL}_2(K)$ acts diagonally from the left on $\mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$. The right hand column arises from the short exact sequence

$$0 \longrightarrow \mathrm{St}_{\mathcal{K}} \xrightarrow{\alpha_{\mathcal{K}}} \mathbb{Z}[\mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}] \xrightarrow{\deg_{\mathcal{K}}} \mathbb{Z}[\mathrm{Cl}_{\mathcal{K}}] \longrightarrow 0,$$

The latter can be derived by induction of groups from the analogous short exact sequences in the local situation. Alternatively, one can prove a global analogue of Proposition 5.16 and give a direct global definition of $\alpha_{\mathcal{K}}$ along the lines of that for α_Γ. ∎

To an adelic harmonic cocycle $\tilde{\mathbf{c}}$ of weight n and level $\mathcal{K}$ over a K-algebra R, we attach the element

$$\Phi_{\mathcal{K}}(\tilde{\mathbf{c}}) := \sum_{(e,g\mathcal{K}) \in \mathrm{GL}_2(K)\backslash \mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}/\{\pm 1\}} [(e, g\mathcal{K})] \otimes \tilde{\mathbf{c}}(e, g) \in \mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}] \otimes_{\mathrm{GL}_2(K)} V_n(R).$$

As in the local case, it is simple to see that $\Phi_{\mathcal{K}}(\tilde{\mathbf{c}})$ lies in fact in the submodule $\mathrm{St}_{\mathcal{K}} \otimes_{\mathrm{GL}_2(K)} V_n(R)$.

Proposition 5.50 *For any K-algebra R, the assignment $\tilde{\mathbf{c}} \mapsto \Phi_{\mathcal{K}}(\tilde{\mathbf{c}})$ defines an isomorphism $\tilde{\mathbf{C}}_n^{\mathrm{har}}(\mathcal{K}, R) \to \mathrm{St}_{\mathcal{K}} \otimes_{\mathrm{GL}_2(K)} V_n(R)$.*

The proposition follows immediately from Propositions 5.11 and 5.40 and the chain of isomorphisms above Definition 5.44.

Combining Theorems 5.19, 5.37 , and Proposition 5.30 with the above proposition yields:

Theorem 5.51 *The isomorphism $\mathbf{S}_n(\mathcal{K}) \cong \mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, \mathbb{C}_\infty)$ given by $\Phi_{\mathcal{K}} \circ \mathrm{Res}_{\mathcal{K}}$ induces an isomorphism between the subspaces $\mathbf{S}_n^2(\mathcal{K})$ and $\mathbf{C}_n^{\mathrm{St},2}(\mathcal{K}, \mathbb{C}_\infty)$.*

6 Hecke operators

We begin this section by abstractly defining a Hecke algebra $\mathcal{H}_{\mathcal{K}}$ corresponding to an open subgroup $\mathcal{K}$ of $\mathrm{GL}_2(\hat{A})$. In a natural way a commutative subalgebra $\mathcal{H}(\mathcal{K},\mathcal{Y})$ is singled out. Furthermore, we also give a local definition of such Hecke algebras avoiding adelic terminology. For these algebras we define an action on modular functions, forms, etc., in their various incarnations, i.e., modular functions of level $\mathcal{K}$, adelic harmonic cocycles and the Steinberg group. At the heart of these definitions lies a geometric interpretation in terms of correspondences, as is well-known in the number theoretic case. Again there is little which is original here. The importance of double cusp forms seems to have been noted first in [20]. It was remarked there that among the spaces of cusp forms with higher vanishing order at the cusps, cf. [7] p. 171, precisely the spaces of modular forms, cusp forms and double cusp forms are stable under the usual Hecke action. Double cusp forms of weight two also played a prominent role in [17].

6.1 Hecke algebras for compact-open subgroups of $\mathrm{GL}_2(\mathbb{A}^f)$

Let $\mathcal{K}$ be a compact open subgroup of $\mathrm{GL}_2(\hat{A})$. We equip the locally compact group $\mathrm{GL}_2(\mathbb{A}^f)$ with a left-invariant Haar measure which is normalized in such a way that $\int_{\mathcal{K}} dg = 1$. Because GL_2 is uni-modular, the measure is also right invariant. Due to the normalization, one has $\int_M dg \in \mathbb{N}$ for any $\mathcal{K}$-bi-invariant compact set M. Therefore we can define for any locally constant compactly supported $\mathcal{K}$-bi-invariant functions F, H on $\mathrm{GL}_2(\mathbb{A}^f)$ with values in A their convolution product $F * G$ by

$$(F * G)(h) := \int F(hg^{-1})G(g)\, dg \quad \forall h \in \mathrm{GL}_2(\mathbb{A}^f).$$

Definition 6.1 *By $\mathcal{H}_{\mathcal{K}}$ we denote the set of compactly supported $\mathcal{K}$-bi-invariant A-valued locally constant functions on $\mathrm{GL}_2(\mathbb{A}^f)$. Under pointwise addition and convolution as multiplication this set is an A-algebra. $\mathcal{H}_{\mathcal{K}}$ is called the* global Hecke algebra for $\mathcal{K}$.

Using the uni-modularity of the chosen Haar-measure, one shows that the convolution operation is associative. Let $\mathbb{1}_{\mathcal{K}x\mathcal{K}}$, $x \in \mathrm{GL}_2(\mathbb{A}^f)$, denote the characteristic function on $\mathcal{K}x\mathcal{K}$. The elements $\mathbb{1}_{\mathcal{K}x\mathcal{K}}$, $x \in \mathrm{GL}_2(\mathbb{A}^f)$, form a basis of $\mathcal{H}_{\mathcal{K}}$. On such basis elements, the convolution operation can be described as follows. Write $\mathcal{K}x\mathcal{K} = \coprod \mathcal{K}x_i$ and $\mathcal{K}y\mathcal{K} = \coprod \mathcal{K}y_j$ as disjoint unions. Then

$$\mathbb{1}_{\mathcal{K}x\mathcal{K}} * \mathbb{1}_{\mathcal{K}y\mathcal{K}} = \sum_{i,j} \mathbb{1}_{\mathcal{K}x_iy_j}. \tag{30}$$

From now on, we will assume that $\mathcal{K} = \prod \mathcal{K}_{\mathfrak{p}}$ where $\mathcal{K}_{\mathfrak{p}}$ is a compact-open subgroup of $\mathrm{GL}_2(A_{\mathfrak{p}})$. Then $\mathcal{H}_{\mathcal{K}}$ is the restricted tensor product $\otimes'_{\mathfrak{p}\in\mathrm{Max}(A)}\mathcal{H}_{\mathcal{K}_{\mathfrak{p}}}$, cf. [6], Ch. III, where $\mathcal{H}_{\mathcal{K}_p}$ is the Hecke algebra of locally constant compactly supported $\mathcal{K}_{\mathfrak{p}}$-bi-invariant A-valued functions on $\mathrm{GL}_2(K_{\mathfrak{p}})$ under addition and convolution. Because $\mathcal{K}$ is compact open, one has $\mathcal{K}_{\mathfrak{p}} = \mathrm{GL}_2(A_{\mathfrak{p}})$ for almost all $\mathfrak{p}$. As one knows that $\mathcal{H}_{\mathrm{GL}_2(A_{\mathfrak{p}})}$ is commutative, the possible non-commutativity of $\mathcal{H}_{\mathcal{K}}$ stems from the finitely many exceptional local factors.

We are not so much interested in $\mathcal{H}_{\mathcal{K}}$ itself, but in a certain commutative subalgebra, which we introduce now, following the treatment given in [52], §2. The basic idea is to restrict the domain of the functions in $\mathcal{H}_{\mathcal{K}_{\mathfrak{p}}}$ at the finitely many exceptional places, so that the subalgebra generated by these functions will be commutative (with respect to the convolution product).

We first define various open subgroups of $\mathrm{GL}_2(K_\mathfrak{p})$. Let $\mathfrak{n}$ be an ideal of A (or $A_\mathfrak{p}$). Let us assume in the following definitions that $x \in \mathrm{GL}_2(A_\mathfrak{p})$ is the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We define

$$\begin{aligned} \mathcal{K}_{0,\mathfrak{p}}(\mathfrak{n}) &:= \{x \in \mathrm{GL}_2(A_\mathfrak{p}) : c \in \mathfrak{n}A_\mathfrak{p}\}, \\ \mathcal{K}_{1,\mathfrak{p}}(\mathfrak{n}) &:= \{x \in \mathcal{K}_{0,\mathfrak{p}}(\mathfrak{n}) : a-1 \in \mathfrak{n}A_\mathfrak{p}\}, \\ \mathcal{K}_{\mathfrak{p}}(\mathfrak{n}) &:= \{x \in \mathcal{K}_{1,\mathfrak{p}}(\mathfrak{n}) : b, d-1 \in \mathfrak{n}A_\mathfrak{p}\}. \end{aligned}$$

In particular, if $\mathfrak{p}$ and $\mathfrak{n}$ are relatively prime then $\mathcal{K}_{?,\mathfrak{p}}(\mathfrak{n}) = \mathrm{GL}_2(A_\mathfrak{p})$.

Definition 6.2 *For* $? \in \{0, 1, \varnothing\}$ *we define* $\mathcal{K}_?(\mathfrak{n}) := \prod_\mathfrak{p} \mathcal{K}_{?,\mathfrak{p}}(\mathfrak{n})$.

Depending on the open subgroup $\mathcal{K}_\mathfrak{p}$ of $\mathrm{GL}_2(A_\mathfrak{p})$, we will now define a $\mathcal{K}_\mathfrak{p}$ bi-invariant semi-subgroup $\mathcal{Y}_\mathfrak{p}$ of $\mathrm{GL}_2(K_\mathfrak{p})$. Recall that $\pi_\mathfrak{p}$ denotes a uniformizer of $A_\mathfrak{p}$. There are three cases:

I) If $\mathcal{K}_\mathfrak{p} = \mathrm{GL}_2(A_\mathfrak{p})$, then we define

$$\mathcal{Y}_\mathfrak{p} := M_2(A_\mathfrak{p}) \cap \mathrm{GL}_2(K_\mathfrak{p}).$$

II) If $\mathcal{K}_\mathfrak{p}$ satisfies $\mathcal{K}_{1,\mathfrak{p}}(\mathfrak{p}^n) \subset \mathcal{K}_\mathfrak{p} \subset \mathcal{K}_{0,p}(\mathfrak{p})$ for some $n \in \mathbb{N}$, then we define

$$\mathcal{Y}_\mathfrak{p} := \bigcup_m \mathcal{K}_\mathfrak{p} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p}^m \end{smallmatrix}\right) \mathcal{K}_\mathfrak{p}.$$

Because $\mathcal{K}_\mathfrak{p}$ contains $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right) : d \in A_\mathfrak{p}^*\}$, this definition is independent of the choice of $\pi_\mathfrak{p}$.

III) Finally, if $\mathcal{K}_\mathfrak{p}$ is of neither of the above types, we define $\mathcal{Y}_\mathfrak{p} := \mathcal{K}_\mathfrak{p}$. According to the above three cases, we call $\mathcal{K}_\mathfrak{p}$ of type I, II and III, respectively.

Note that if $\mathcal{K}$ is admissible, then not all factors can be of type I or II. Finally, we define $\mathcal{Y}$ as the restricted product of the $\mathcal{Y}_\mathfrak{p}$ relative to the $\mathcal{K}_\mathfrak{p}$. Then $\mathcal{Y}$ is a semi-subgroup of $\mathrm{GL}_2(\mathbb{A}^f)$.

Definition 6.3 *Let* $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ *denote the subalgebra of* $\mathcal{H}_\mathcal{K}$ *of functions whose support lies in* $\mathcal{Y}$. *We abbreviate* $\mathcal{H}(\mathcal{K}_?(\mathfrak{n}), \mathcal{Y})$ *by* $\mathcal{H}_?(\mathfrak{n})$.

Remark 6.4 Regarding the local factor $\mathcal{Y}_\mathfrak{p}$ for a group $\mathcal{K}_\mathfrak{p}$ of type II, there are various things to note.

(i) $\mathcal{K}_\mathfrak{p}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{smallmatrix}\right)\mathcal{K}_\mathfrak{p}$ is the disjoint union $\coprod_{b \in k_\mathfrak{p}} \mathcal{K}_\mathfrak{p}\left(\begin{smallmatrix} 1 & b \\ 0 & \pi_\mathfrak{p} \end{smallmatrix}\right)$.

(ii) $\mathcal{K}_\mathfrak{p}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p}^m \end{smallmatrix}\right)\mathcal{K}_\mathfrak{p} = \left(\mathcal{K}_\mathfrak{p}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{smallmatrix}\right)\mathcal{K}_\mathfrak{p}\right)^m$.

(iii) Let $T_{\mathfrak{p}^m}$ denote $\mathbb{1}_{\mathcal{K}_\mathfrak{p}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p}^m \end{smallmatrix}\right)\mathcal{K}_\mathfrak{p}}$. Then one has $T_{\mathfrak{p}^m} * T_{\mathfrak{p}^{m'}} = T_{\mathfrak{p}^{m+m'}}$ for the convolution product.

To define a generating set for the Hecke algebras $\mathcal{H}(\mathcal{K}, \mathcal{Y})$, we choose for each fractional ideal $\mathfrak{m}$ of A an element $a_\mathfrak{m} \in (\mathbb{A}^f)^*$ such that $a_\mathfrak{m}\hat{A} = \mathfrak{m}\hat{A}$ in $\mathbb{A}^f$ and such that all components $(a_\mathfrak{m})_\mathfrak{p} = 1$ for $\mathfrak{p} \not| \mathfrak{m}$.

Definition 6.5 *Let* $\mathfrak{m}$ *be any ideal of* A. *By* $T_{\mathfrak{m}}(\mathcal{K}) \in \mathcal{H}(\mathcal{K}, \mathcal{Y})$ *we denote the characteristic function on the compact* $\mathcal{K}$*-bi-invariant subset*

$$\{y \in \mathcal{Y} : \det(y) = \mathfrak{m}\}.$$

Furthermore, for any ideal $\mathfrak{m}$ *of* A *we define* $S_{\mathfrak{m}}(\mathcal{K}) \in \mathcal{H}(\mathcal{K}, \mathcal{Y})$ *by*

$$S_{\mathfrak{m}}(\mathcal{K}) := \begin{cases} 0 & \text{if } \mathfrak{m} \text{ is not prime to the minimal conductor } \mathfrak{n} \text{ of } \mathcal{K} \\ \mathbb{1}_{a_{\mathfrak{m}}\mathcal{K}} & \text{if } \mathfrak{m} \text{ is prime to } \mathfrak{n} \end{cases}$$

It is easy to see that $S_{\mathfrak{m}}(\mathcal{K})$ is independent of the choice of $a_{\mathfrak{m}}$. When there is no fear of confusion, we simply write $S_{\mathfrak{m}}$ and $T_{\mathfrak{m}}$ instead of $S_{\mathfrak{m}}(\mathcal{K})$ and $T_{\mathfrak{m}}(\mathcal{K})$. Note that if $\mathfrak{m}$ contains a prime $\mathfrak{p}$ in its product expansion such that $\mathcal{K}_{\mathfrak{p}}$ is of type III, then $T_{\mathfrak{m}}(\mathcal{K}) = 0$.

As the elements of $\mathcal{H}_{\mathcal{K}}$ have coefficients in a field of characteristic p, the usual relation for the product $T_{\mathfrak{m}}T_{\mathfrak{n}}$ implies that

$$T_{\mathfrak{m}}T_{\mathfrak{n}} = T_{\mathfrak{mn}} \quad \text{for any ideals } \mathfrak{m}, \mathfrak{n}.$$

This is strikingly different from the number field case, where the above only holds for relatively prime ideals.

Proposition 6.6 *The Hecke algebra* $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ *is commutative and generated by the elements* $T_{\mathfrak{m}}(\mathcal{K})$ *and* $S_{\mathfrak{m}}(\mathcal{K})$ *where* $\mathfrak{m}$ *runs through all ideals of* A.

PROOF: For the commutativity assertion, it suffices to consider the local components above all non-zero primes $\mathfrak{p}$ of A. If $\mathcal{K}_{\mathfrak{p}}$ is of type I, this is well-known, cf. [6], Ch. IV. If $\mathcal{K}_{\mathfrak{p}}$ is of type II, this was noted in Remark 6.4. In the remaining case, this is trivial.

To see that the $T_{\mathfrak{m}}$ and $S_{\mathfrak{m}}$ form a generating set, it suffices to show that the local Hecke algebra above $\mathfrak{p}$ is generated as a A-algebra by $T_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$. Again, this follows from [6], Ch. IV, or Remark 6.4 or is trivial, respectively. ∎

As with the description of modular forms either as $h_{\mathcal{K}} = \operatorname{card} \mathrm{Cl}_{\mathcal{K}}$-tuples of holomorphic functions on quotients $\Gamma_\nu \backslash \Omega$ or as functions on $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A})/\mathcal{K}$, the above adelic description of the Hecke algebras $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ has a 'local' counterpart in terms of double cosets for the arithmetic subgroups Γ_ν of $\mathrm{GL}_2(K)$.

For $\mu, \nu \in \mathrm{Cl}_{\mathcal{K}}$, define

$$R_{\mu,\nu}(\mathcal{K}) := A[\Gamma_\mu \alpha \Gamma_\nu : \alpha \in \mathrm{GL}_2(K) \cap x_\mu \mathcal{Y} x_\nu^{-1}],$$

i.e., as the free A-module on certain double cosets of $\Gamma_\mu\backslash\mathrm{GL}_2(K)/\Gamma_\nu$. Note that any double coset $\Gamma_\mu\alpha\Gamma_\nu$ can be written as a disjoint union $\coprod \Gamma_\mu \alpha_i$ of left cosets of $\Gamma_\mu\backslash\mathrm{GL}_2(K)$. Define $R(\mathcal{K}, \mathcal{Y})$ as the direct sum over all $R_{\mu,\nu}(\mathcal{K})$, where one should think of this as some kind of matrix algebra due to the following definition of multiplication for elements in $R(\mathcal{K}, \mathcal{Y})$:

The product between elements of $R_{\mu,\nu}(\mathcal{K})$ and $R_{\mu',\nu'}(\mathcal{K})$ is zero unless $\nu = \mu'$ and in this case it yields elements in $R_{\mu,\nu'}(\mathcal{K})$. For $\Gamma_\lambda\alpha\Gamma_\mu$ and $\Gamma_\mu\beta\Gamma_\nu$ choose elements α_i and β_j such that

$$\Gamma_\lambda\alpha\Gamma_\mu = \coprod \Gamma_\lambda\alpha_i \quad \text{and} \quad \Gamma_\mu\beta\Gamma_\nu = \coprod \Gamma_\mu\beta_j.$$

Then the multiplication $R_{\lambda,\mu}(\mathcal{K}) \times R_{\mu,\nu}(\mathcal{K}) \to R_{\lambda,\nu}(\mathcal{K})$ is defined by

$$[\Gamma_\lambda\alpha\Gamma_\mu] \cdot [\Gamma_\mu\beta\Gamma_\nu] = \sum m(\alpha, \beta, \gamma)[\Gamma_\lambda\gamma\Gamma_\nu], \tag{31}$$

where the sum is over all double cosets $\Gamma_\lambda\gamma\Gamma_\nu$ such that $\coprod \Gamma_\lambda\gamma\Gamma_\nu = \Gamma_\lambda\alpha\Gamma_\mu\beta\Gamma_\nu$ and the $m(\alpha,\beta,\gamma) \in \mathbb{F}_{\mathbb{F}_p}$ are defined as the reduction mod p of

$$\mathrm{card}\{(i,j) : \Gamma_\lambda\alpha_i\beta_j = \Gamma_\lambda\gamma\}.$$

One may verify that this definition is independent of any choices made and that it defines an associative product on $R(\mathcal{K},\mathcal{Y})$.

Finally we define a map $\xi\colon \mathcal{H}(\mathcal{K},\mathcal{Y}) \to R(\mathcal{K},\mathcal{Y})$ as follows. For $y \in \mathcal{Y}$ and $\mu \in \mathrm{Cl}_\mathcal{K}$ write $x_\nu y^{-1} = \alpha_{y,\mu}^{-1} x_\mu g \in \mathrm{GL}_2(K)x_\mu\mathcal{K}$ for some uniquely determined $\nu = \nu_{y,\mu}$, which depends on y and μ, and set

$$\xi(\mathbb{1}_{\mathcal{K}y\mathcal{K}}) := \sum_\mu [\Gamma_\mu\alpha_{y,\mu}\Gamma_{\nu_{y,\mu}}]. \tag{32}$$

Again this is independent of the chosen $\alpha_{y,\mu}$.

Proposition 6.7 *The map $\xi\colon \mathcal{H}(\mathcal{K},\mathcal{Y}) \to R(\mathcal{K},\mathcal{Y})$ is an injective ring homomorphism between commutative A-algebras.*

Proof: We will only show that ξ is compatible with taking products and leave the proofs of the other assertions to the reader. This we may verify on elements of the form $\mathbb{1}_{\mathcal{K}y\mathcal{K}}$, $y \in \mathcal{Y}$.

For such an element, write $\mathcal{K}y\mathcal{K} = \coprod \mathcal{K}y_j$ and define for each j and μ elements $\nu_\mu \in \mathrm{Cl}_\mathcal{K}$ and elements $\alpha_{\mu,j} \in \mathrm{GL}_2(K)$ such that

$$x_{\nu_\mu} y_j^{-1} \in \alpha_{\mu,j}^{-1} x_\mu \mathcal{K}.$$

Note that ν_μ is determined by the requirement that $t_\mu \det(y) t_{\nu_\mu}^{-1} \in K^* \det \mathcal{K}$. Let j_0 be any of the j. Then it is easy to see that

$$\Gamma_\mu\alpha_{\mu,j_0}\Gamma_\nu = \coprod \Gamma_\mu\alpha_{\mu,j}.$$

From equations (30) and (31), which describe the multiplication on basis elements for $\mathcal{H}_\mathcal{K}$ and $R(\mathcal{Y},\mathcal{K})$, respectively, one can now see the asserted compatibility of ξ with multiplication. ∎

6.2 Hecke operators on modular forms

We now define an action of $\mathcal{H}(\mathcal{K},\mathcal{Y})$ on spaces of modular forms of level $\mathcal{K}$. Fix $y \in \mathrm{GL}_2(\mathbb{A}_f)$ and choose elements $y_j \in \mathrm{GL}_2(\mathbb{A}_f)$ such that

$$\mathcal{K}y\mathcal{K} = \amalg \mathcal{K}y_j. \tag{33}$$

From now on, given $y \in \mathcal{Y}$, elements y_j will always denote left coset representatives of $\mathcal{K}y\mathcal{K}$.

Definition 6.8 *For a modular form $\mathbf{f}$ the operation of $\mathbb{1}_{\mathcal{K}y\mathcal{K}}$ is defined as*

$$\mathbf{f} \mapsto \mathbf{f}_{|\mathcal{K}y\mathcal{K}} : (z, w_\mathrm{f}, w_\infty) \mapsto \sum_j \mathbf{f}(z, w_\mathrm{f} y_j^{-1}, w_\infty).$$

If $\mathcal{K}$ is the level of $\mathbf{f}$, then the individual functions $\mathbf{f}(z, w_\mathrm{f} y_j^{-1}, w_\infty)$ are invariant under $\mathrm{GL}_2(K) \times y_j\mathcal{K}y_j^{-1}$. A simple computation, which is left to the reader, shows that $\mathbf{f}_{|\mathcal{K}y\mathcal{K}}$ is again invariant under $\mathrm{GL}_2(K) \times \mathcal{K}$ and moreover an adelic modular form of level $\mathcal{K}$.

Proposition 6.9 *(i) For $y, y' \in \mathcal{Y}$, the operation $\mathbf{f} \mapsto \mathbf{f}_{|\mathcal{K}y\mathcal{K}}$ satisfies*

$$(\mathbf{f}_{|\mathcal{K}y\mathcal{K}})_{|\mathcal{K}y'\mathcal{K}} = \sum_j \mathbf{f}_{|\mathcal{K}y''_j\mathcal{K}} \tag{34}$$

if the elements y''_j are defined so that $\mathbb{1}_{\mathcal{K}y\mathcal{K}} * \mathbb{1}_{\mathcal{K}y'\mathcal{K}} = \sum_j \mathbb{1}_{\mathcal{K}y''_j\mathcal{K}}$. In particular $\mathbf{f} \mapsto \mathbf{f}_{|\mathcal{K}y\mathcal{K}}$ extends linearly to a right action of $\mathcal{H}_{\mathcal{K}}$ on the space of modular functions of weight n, type l and level $\mathcal{K}$.

(ii) The operation defined in (i) preserves the subspaces of modular forms, cusp forms and double cusp forms.

(iii) Furthermore for $\mathcal{K}' \subset \mathcal{K}$, the actions of $\mathcal{H}(\mathcal{K}', \mathcal{Y}') \subset \mathcal{H}(\mathcal{K}, \mathcal{Y})$ on $\mathbf{S}_{n,l}(\mathcal{K})$ and on the image of $\mathbf{S}_{n,l}(\mathcal{K})$ in $\mathbf{S}_{n,l}(\mathcal{K}')$ agree.

Before we prove this, we first recall a geometric interpretation of the Hecke operator attached to $\mathcal{K}y\mathcal{K}$. Consider

$$\begin{array}{ccc} \mathfrak{X}_{\mathcal{K}\cap y\mathcal{K}y^{-1}} & \xrightarrow{r_y} & \mathfrak{X}_{\mathcal{K}\cap y^{-1}\mathcal{K}y} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathfrak{X}_{\mathcal{K}} & & \mathfrak{X}_{\mathcal{K}}, \end{array} \tag{35}$$

where the π_i are the natural projections that arise from the inclusions $\mathcal{K} \cap y^{-1}\mathcal{K}y, \mathcal{K}\cap y\mathcal{K}y^{-1} \subset \mathcal{K}$, and where r_y is right multiplication by $y \in \mathrm{GL}_2(\mathbb{A}^f)$. By π_1^* and $r_{y^{-1}}^*$ we denote the pullback of functions under π_1 and $r_{y^{-1}}$, respectively. The map trace_{π_2} denotes the trace of the pushforward, i.e., trace_{π_2} assigns to a function $\mathbf{f}$ on $\mathfrak{X}_{\mathcal{K}\cap y^{-1}\mathcal{K}y}$ the function on $\mathfrak{X}_{\mathcal{K}}$ whose value at a point x is the sum of the values of $\mathbf{f}$ at all preimages of x. Then

$$\mathbf{f}_{|\mathcal{K}y\mathcal{K}} = \mathrm{trace}_{\pi_2} r_{y^{-1}}^* \pi_1^* \mathbf{f} = \mathrm{trace}_{\pi_2 r_y} \pi_1^* \mathbf{f}.$$

Proof: Formula (34) easily follows from (30). That the operation $\mathbf{f} \mapsto \mathbf{f}_{|\mathcal{K}y\mathcal{K}}$ extends linearly to $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ is then easy to verify. We leave part (iii) to the reader and now turn to the proof of part (ii).

To see that the operation of an element $\mathcal{K}y\mathcal{K}$ preserves the subspaces of modular forms, cusp forms and double cusp forms, it suffices to show this for each of the maps $(\pi_1 r_{y^{-1}})^*$, and trace_{π_2}. For the first, this is obvious. Thus we now consider $\mathrm{trace}_{\pi_2}\mathbf{f}$, and we let $r \in \{0, 1, 2\}$ denote the vanishing order of $\mathbf{f}$ at the cusps.

Let c be a cusp of $\mathfrak{X}_{\mathcal{K}}$ and let $c_1, \ldots, c_s$ be the cusp above c under the map π_2. Let R be the local ring at c and R_i that at c_i for $i = 1, \ldots, r$. Let $\mathbf{f}_i \in R_i$ be the restriction of $\mathbf{f}$ to R_i and trace_i the trace map from R_i to R. Then $\mathrm{trace}_{\pi_2}\mathbf{f} = \sum_i \mathrm{trace}_i \mathbf{f}_i$ when restricted to R. Therefore it suffices to verify the claim for each of the maps trace_i.

These trace maps are the usual trace maps for a finite flat morphism between local rings. Furthermore, the maps $R \to R_i$ are totally wildly ramified and Galois. In fact, the proof given in [15], VII.5, shows that the second higher ramification group of $R \to R_i$ is trivial and that the first higher ramification group is the whole decomposition group at c_i. Therefore the assertion is a direct consequence of the following lemma. ∎

Lemma 6.10 *Let $\pi : R \to R'$ be a Galois cover between complete discrete valuation rings of characteristic p, which is totally wildly ramfied. Let $\mathfrak{m}$ and $\mathfrak{m}'$ be the respective maximal ideals. Assume that the second higher ramification group is trivial and the first higher ramification group is the whole Galois group. Then* $\operatorname{trace}(\mathfrak{m}'^r) \subset \mathfrak{m}^r$ *for $r = 0, 1, 2$.*

Proof: For $r = 0, 1$, the assertion is trivial. So we now consider the case $r = 2$. Let G_i denote the i-th higher ramification group (in lower indexing). Then by [50], one has $\mathfrak{D}_{R'/R} = \mathfrak{m}'^t$ for the different of R'/R, where $t = \sum_{i=0}^{\infty}(|G_i| - 1)$. In the situation given, this yields $\mathfrak{D}_{R'/R} = \mathfrak{m}'^{2(|G|-1)}$, where G is the Galois group of R'/R. Because the ring extension is totally wildly ramified, one has $\mathfrak{m} = \mathfrak{m}'^{|G|}$, so that $\mathfrak{m}^2 = \mathfrak{D}_{R/R'}\mathfrak{m}'^2$. The above and the definition of the discriminant now implies that

$$\operatorname{trace}(\mathfrak{m}'^2) = \operatorname{trace}(\mathfrak{m}^2\mathfrak{D}_{R/R'}^{-1}) = \mathfrak{m}^2\operatorname{trace}(\mathfrak{D}_{R/R'}^{-1}) \subset \mathfrak{m}^2,$$

as claimed. ∎

Remark 6.11 The above short proof we owe to R. Pink. Alternatively, one can use the theory of Goss polynomials and an explicit computation to prove the Proposition. These explicit computations can also be used to show that triple cusp forms are not preserved under Hecke operators.

This is strikingly different from the classical case. There the local ramification at the cusps is tame, and one can show that double cusp forms are not preserved under the Hecke operators.

Remark 6.12 For $y \in \mathrm{GL}_2(\mathbb{A}^f)$, choose $\nu \in \mathrm{Cl}_{\mathcal{K}}$ and $\alpha \in \mathrm{GL}_2(K)$ such that $y \in \alpha x_\nu \mathcal{K}$. Let l and l' be congruent modulo l_Γ. Let $\mathbf{f}$ and $\mathbf{f}'$ be adelic modular functions of level $\mathcal{K}$ weight n and type l and l', respectively, such that $\mathbf{f} \mapsto \mathbf{f}'$ under the isomorphism of Lemma 5.32. Then under the same isomorphism one has

$$\mathbf{f}_{|\mathcal{K}y\mathcal{K}} \mapsto \det \alpha^{l-l'} \mathbf{f}'_{|\mathcal{K}y\mathcal{K}}.$$

Thus while the spaces $\underline{M}_{n,l'}(\mathcal{K})$, $\underline{S}_{n,l'}(\mathcal{K})$, etc., only depend on l' modulo $l_{\mathcal{K}}$, the Hecke actions defined above depend on $l \in \mathbb{Z}$. However the eigenfunctions for the action of $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ only depend on $l \pmod{l_{\mathcal{K}}}$.

To be able to compare our definition with the more classical one, given in [15], VIII, or [16], §7, we translate the above definition to modular forms given in their local description. So suppose that $\mathbf{f}$ corresponds to the tuple of functions $\underline{f} = \{f_\nu\}_{\nu \in \mathrm{Cl}_{\mathcal{K}}}$. We want to describe the local components of $\mathbf{f}_{|\mathcal{K}y\mathcal{K}}$, which we denote by $(\underline{f}_{|\mathcal{K}y\mathcal{K}})_\nu$, $\nu \in \mathrm{Cl}_{\mathcal{K}}$.

Fix ν and $y \in \mathrm{GL}_2(\mathbb{A}^f)$. This determines a unique $\mu \in \mathrm{Cl}_{\mathcal{K}}$ such that $x_\nu y^{-1} \in \mathrm{GL}_2(K)x_\mu\mathcal{K}$. Furthermore, choose elements α and α_j in $\mathrm{GL}_2(K)$ such that $x_\nu y^{-1} \in \alpha^{-1}x_\mu\mathcal{K}$ and $x_\nu y_j^{-1} \in \alpha_j^{-1}x_\mu\mathcal{K}$. By formula (25) and using Definition 6.8 and the definition of $\mathbf{f}$ in terms of $\underline{f}$ on page 60 we have

$$(\underline{f}_{|\mathcal{K}y\mathcal{K}})_\nu = \sum_j f_\mu \underset{n,l}{||} \alpha_j. \tag{36}$$

If desired, one could use the dictionary between the local and global viewpoint to directly define an action for a class $[\Gamma_\mu\alpha\Gamma_\nu]$ of $R(\mathcal{K}, \mathcal{Y})$ on tuples $\underline{f}$ and derive from this the above definition of $\mathbf{f}_{|\mathcal{K}y\mathcal{K}}$.

Example 6.13 Suppose $A = k[T]$ and $\mathcal{K} = \mathcal{K}((T))$. Then $\mathrm{Cl}_{\mathcal{K}}$ is trivial. Given a maximal ideal $\mathfrak{p} \neq (T)$, let $p_{\mathfrak{p}}$ be a generator of $\mathfrak{p}$ such that $p_{\mathfrak{p}} \equiv 1 \pmod T$ and consider $y = \begin{pmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}$. To compute the action of $T_{\mathfrak{p}} = \underline{}_{|\mathcal{K}y\mathcal{K}}$, let $E_{\mathfrak{p}}$ be the set of polynomials of degree less than $\deg \mathfrak{p}$. Then

$$\mathcal{K}y\mathcal{K} = \mathcal{K}\begin{pmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \amalg \coprod_{b \in E_{\mathfrak{p}}} \mathcal{K}\begin{pmatrix} 1 & \frac{b(1-p_{\mathfrak{p}})}{p_{\mathfrak{p}}} \\ 0 & \end{pmatrix}.$$

We use $p_{\mathfrak{p}} \equiv 1 \pmod T$ since this implies that

$$\begin{pmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & \frac{b(1-p_{\mathfrak{p}})}{p_{\mathfrak{p}}} \\ 0 & \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod T,$$

and therefore can take for the α_j the elements

$$\begin{pmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \frac{b(1-p_{\mathfrak{p}})}{p_{\mathfrak{p}}} \\ 0 & \end{pmatrix}$$

Thus we find

$$\begin{aligned} T_{\mathfrak{p}} f(z) &= (\underline{f}_{|\mathcal{K}y\mathcal{K}})(z) \\ &= \Big(f \mathop{||}_{n,l} \begin{pmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\Big)(z) + \sum_{b \in E_{\mathfrak{p}}} \Big(f \mathop{||}_{n,l} \begin{pmatrix} 1 & \frac{b(p_{\mathfrak{p}}^{-1}-1)}{p_{\mathfrak{p}}} \\ 0 & \end{pmatrix}\Big)(z) \\ &= p_{\mathfrak{p}}^{l-n}\Big(p_{\mathfrak{p}}^{n} f(p_{\mathfrak{p}} z) + \sum_{b \in E_{\mathfrak{p}}} f((z + b(1-p_{\mathfrak{p}}))/p_{\mathfrak{p}})\Big). \end{aligned}$$

Suppose now further that $f \in \mathbf{S}_{n,l}(\mathrm{GL}_2(\hat{A}))$. Then the last line simplifies to

$$p_{\mathfrak{p}}^{l-n}\Big(p_{\mathfrak{p}}^{n} f(p_{\mathfrak{p}} z) + \sum_{b \in E_{\mathfrak{p}}} f((z+b)/p_{\mathfrak{p}})\Big).$$

By Proposition 6.9 (iii), the Hecke operator $T_{\mathfrak{p}}$ can also be computed via the coset decomposition

$$\mathrm{GL}_2(\hat{A})\begin{pmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\mathrm{GL}_2(\hat{A}) = \mathrm{GL}_2(\hat{A})\begin{pmatrix} p'_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \amalg \coprod_{b \in E_{\mathfrak{p}}} \mathrm{GL}_2(\hat{A})\begin{pmatrix} 1 & b \\ 0 & p'_{\mathfrak{p}} \end{pmatrix},$$

where $p'_{\mathfrak{p}}$ is the monic generator of $\mathfrak{p}$. If we furthermore use $l = n-1$, we find that

$$T_{\mathfrak{p}} f(z) = {p'_{\mathfrak{p}}}^{-1}\Big({p'_{\mathfrak{p}}}^{n} f(p'_{\mathfrak{p}} z) + \sum_{b \in E_{\mathfrak{p}}} f((z+b)/p'_{\mathfrak{p}})\Big).$$

This can now be directly compared with the Hecke operator at $\mathfrak{p}$ defined in [16], (7.1), which we denote by $T'_{\mathfrak{p}}$. We take $l = q$ and $n = q+1$. Then on $\mathbf{S}_{q+1}(\mathcal{K}((T)))$ one has

$$T_{\mathfrak{p}} f = (p'_{\mathfrak{p}})^{-1} T'_{\mathfrak{p}} f.$$

This means that one has the same Hecke eigenforms, however to different systems of Hecke eigenvalues.

6.3 Hecke operators on harmonic cocycles

The issue here is to translate the above definition of a Hecke operation on adelic modular forms to an action on harmonic cocycles which is compatible with the residue map. We fix a K_∞-algebra R and a K-algebra $\tilde{R}$.

Definition 6.14 *For* $\mathbf{c}$ *in* $\mathbf{C}^{\text{har}}_{n,l}(\mathcal{K}, R)$, $\tilde{\mathbf{c}}$ *in* $\tilde{\mathbf{C}}^{\text{har}}_{n,l}(\mathcal{K}, \tilde{R})$ *and* y *in* $\mathcal{Y}$ *we define* $\mathbf{c}_{|\mathcal{K}y\mathcal{K}}$ *and* $\tilde{\mathbf{c}}_{|\mathcal{K}y\mathcal{K}}$ *by:*

$$\begin{aligned} \mathbf{c}_{|\mathcal{K}y\mathcal{K}}(e, w_f, w_\infty) &:= \sum_j \mathbf{c}(e, w_f y_j^{-1}, w_\infty), \\ \tilde{\mathbf{c}}_{|\mathcal{K}y\mathcal{K}}(e, w_f) &:= \sum_j \tilde{\mathbf{c}}(e, w_f y_j^{-1}). \end{aligned}$$

Furthermore, for $\underline{c} := (c_\nu) \in \amalg_\nu C^{\text{har}}_{n,l}(\Gamma_\nu, \tilde{R})$ *corresponding to* $\tilde{\mathbf{c}}$*, one defines*

$$(\underline{c}_{|\mathcal{K}y\mathcal{K}})_\nu(e) := \tilde{\mathbf{c}}_{|\mathcal{K}y\mathcal{K}}(e, x_\nu) = \sum_j (\alpha_j^{-1})_{n,l} c_\mu(\alpha_j e),$$

if one has chosen α_j *such that* $x_\nu y_j^{-1} \in \alpha_j^{-1} x_\mu \mathcal{K}$.

Using the isomorphism in Theorem 5.37 and the bijection in Lemma 5.38, it is a simple matter to verify the following proposition:

Proposition 6.15 *The isomorphisms*

$$\mathbf{S}_{n,l}(\mathcal{K}) \xrightarrow{\cong} \mathbf{C}^{\text{har}}_{n,l}(\mathcal{K}, \mathbb{C}_\infty) : \mathbf{f} \mapsto \operatorname{Res} \mathbf{f},$$

$$\mathbf{C}^{\text{har}}_{n,l}(\mathcal{K}, R) \xrightarrow{\cong} \tilde{\mathbf{C}}^{\text{har}}_{n,l}(\mathcal{K}, R) : \mathbf{c} \mapsto \tilde{\mathbf{c}} \text{ and}$$
$$\tilde{\mathbf{C}}^{\text{har}}_{n,l}(\mathcal{K}, \tilde{R}) \to \amalg_\nu C^{\text{har}}_{n,l}(\Gamma_\nu, \tilde{R})$$

are Hecke-equivariant.

Example 6.16 We continue with Example 6.13 in order to express the Hecke operation on harmonic cocycles given in their local description. So we assume the set-up from loc. cit. Let c be the cocycle corresponding to f. The formula at the end of Definition 6.14 together with formula (17), after Definition 5.9, yields

$$\begin{aligned} &p_{\mathfrak{p}}^{l+1-n}(c_{|\mathcal{K}y\mathcal{K}})(e) \\ &= \begin{pmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}^{-1} c\left(\begin{pmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} e\right) + \sum_{b \in E_{\mathfrak{p}}} \begin{pmatrix} 1 & b(1-p_{\mathfrak{p}}) \\ 0 & p_{\mathfrak{p}} \end{pmatrix}^{-1} c\left(\begin{pmatrix} 1 & b(1-p_{\mathfrak{p}}) \\ 0 & p_{\mathfrak{p}} \end{pmatrix} e\right). \end{aligned}$$

This may be used to compute the Hecke action on harmonic cocycles, as well as eigenvalues and eigenforms, if one is in the situation of Example 6.13.

6.4 Hecke operators on the Steinberg module

Our next aim is to give an action of $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ on the Steinberg module $\mathrm{St}_{\mathcal{K}}$, compatible with the action on harmonic cocycles. As noted in the proof of [55], inequality (3), pp. 504ff., while it is true that a harmonic cocycle is completely determined by its values on the stable edges, its values on unstable edges are in general not all zero. More precisely, the value of a cocycle on an unstable edge is determined by the values on its 'stable sources'. This observation is important for the action given by $y \in \mathcal{Y}$, cf. Diagram (35), because the set of stable simplices for $\mathcal{K} \cap y^{-1}\mathcal{K}y$ is larger as that for $\mathcal{K}$. Therefore, below we need to recall the concept of sources.

In the following $\mathcal{K}$ will be admissible, so that all Γ_ν are p'-torsion free. We freely use the notation introduced when describing modular forms via the Steinberg module. Also recall that given an arithmetic p'-torsion free group Γ, below Definition 3.22 we had defined a map $b : \mathcal{T}_\infty \to \mathbb{P}^1(K)$ by sending a simplex t to $\varepsilon_t[\underline{s}]$ where $[\underline{s}]$ is the unique end such that $\Gamma_t \subset \Gamma_{\underline{s}}$, ε_t is 1 if t is a vertex or an edge pointing towards $[\underline{s}]$, and -1 otherwise. We define b_ν as the corresponding map from the unstable simplices of $\mathcal{T}_\nu$ to the ends of $\mathcal{T}_\nu$. By $\mathbf{b}$ we denote the induced map from unstable simplices of $\mathcal{T}_\mathcal{K}$ to the rational ends $\mathbb{P}^1(K)_\mathcal{K} := \mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ of $\mathcal{T}_\mathcal{K}$. One can directly characterize $\mathbf{b}$, by saying that it maps a simplex $\tilde{t} := (t, g\mathcal{K})$ to $\varepsilon_{\tilde{t}}[\tilde{\underline{s}}]$ where $[\tilde{\underline{s}}] = ([\underline{s}], g\mathcal{K}) \in \mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ is the unique element such that for the action of $\mathrm{GL}_2(K)$, the stabilizer of $\tilde{t}$ is contained in the stabilizer of $[\tilde{\underline{s}}]$, and such that $\varepsilon_{\tilde{t}}$ is in $\{\pm 1\}$ and determined in the same way as ε_t above.

Definition 6.17 *For an edge $\tilde{e}$ of $\mathcal{T}^o_\mathcal{K}$, define its source, which is a subset of $\mathcal{T}^o_{\mathcal{K},1}$, as follows: If $\tilde{e}$ is stable, then $\mathrm{src}(\tilde{e}) = \{\tilde{e}\}$. If $\tilde{e}$ is unstable, then $\mathrm{src}(\tilde{e})$ is the set of all $\tilde{e}' \in \mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}$ such that*

(i) there exists an unstable vertex $\tilde{v}'$ of $\tilde{e}'$ such that $\tilde{e}$ lies on the half line from $\tilde{v}'$ to $\mathbf{b}(\tilde{v}')$ and

(ii) $\tilde{e}'$ has the same orientation as $\tilde{e}$ (along that half line).

For $\tilde{e} \in \mathcal{T}^{\mathrm{st}}_{\mathcal{K},1}$ define its course *as*

$$\mathrm{crs}(\tilde{e}) := \{\tilde{e}' \in \mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1} : \tilde{e} \in \mathrm{src}(\tilde{e}')\}.$$

Note that $\mathrm{crs}(\tilde{e})$, $\mathrm{src}(\tilde{e})$ and $\tilde{e}$ lie in the same connected component of $\mathcal{T} \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$.

Translating Lemma 5.13, which is taken from [55], into an adelic context, one obtains the following:

Lemma 6.18 *Let $\tilde{\mathbf{c}}$ be an element of $\tilde{\mathbf{C}}^{\mathrm{har}}_{n,l}(R)$. Then for all $\tilde{e} \in \mathcal{T}^o_{\mathcal{K},1}$ one has*

$$\tilde{\mathbf{c}}(\tilde{e}) = \sum_{\tilde{e}' \in \mathrm{src}(\tilde{e})} \tilde{\mathbf{c}}(\tilde{e}') = \sum_{\tilde{e} \in \mathrm{crs}(\tilde{e}')} \tilde{\mathbf{c}}(\tilde{e}').$$

To define a Hecke action on $\mathrm{St}_\mathcal{K}$, we first define actions on $\mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},0}]$ and on $\mathbb{Z}[\mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}]$. For this we choose elements y'_j such that

$$\coprod y'_j\mathcal{K} = \mathcal{K}y\mathcal{K}. \tag{37}$$

Definition 6.19 *For $y \in \mathcal{Y}$, $(v, g\mathcal{K}) \in \mathcal{T}^{\mathrm{st}}_{\mathcal{K},0}$ and $\tilde{e} \in \mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}$ define*

$$[(v, g\mathcal{K})]_{|\mathcal{K}y\mathcal{K}} := \begin{cases} \sum_j [(v, gy'_j\mathcal{K})] & \text{for } (v, g\mathcal{K}) \in \mathcal{T}^{\mathrm{st}}_0, \\ 0 & \text{otherwise.} \end{cases} \in \mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},0}]$$

$$[\tilde{e}]_{|\mathcal{K}y\mathcal{K}} := \sum_{(e', g'\mathcal{K}) \in \mathrm{crs}(\tilde{e})} \Big(\sum_j [(e', g'y'_j\mathcal{K})]\Big) \in \mathbb{Z}[\mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}],$$

where $[(e', g'y'_j\mathcal{K})] = 0$ whenever $(e', g'y'_j\mathcal{K})$ is unstable.

Let us first explain why the sum defining $[\tilde{e}]_{|\mathcal{K}y\mathcal{K}}$ is finite. For this, we fix j. The elements $(e', g'\mathcal{K})$ in $\mathrm{crs}(\tilde{e})$ form

(i) the one element set $\{\tilde{e}\}$, if both vertices of $\tilde{e}$ are stable,

(ii) a half line starting at $\tilde{v}_0$ whose equivalence class is a rational end, if precisely one vertex of $\tilde{e}$, namely $\tilde{v}_0$, is unstable,

(iii) a line from one to another rational end which contains $\tilde{e}$, if both vertices of $\tilde{e}$ are unstable.

Therefore the elements $\{(e, gy'_j\mathcal{K}) : (e', g'\mathcal{K}) \in \mathrm{crs}(\tilde{e})\}$ form a translate of either of the above sets, of precisely the same shape. But each half line going to a rational end contains only a finite number of stable edges, and thus the sum is finite as asserted.

The following result is obvious and left to the reader:

Lemma 6.20 *The action $|_{\mathcal{K}y\mathcal{K}}$ on $\mathbb{Z}[\mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}]$ induces an action on $\mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}]$.*

The induced action is again denoted by $[\tilde{e}] \mapsto [\tilde{e}]_{|\mathcal{K}y\mathcal{K}}$. By linearity, the above definitions extend to left actions of $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ on $\mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},0}]$, $\mathbb{Z}[\mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}]$ and $\mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}]$. The next lemma shows that these are compatible with the boundary map $\partial_{\mathcal{K}}$:

Lemma 6.21 *For $m \in \mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}]$ one has*

$$\partial_{\mathcal{K}}(m_{|\mathcal{K}y\mathcal{K}}) = (\partial_{\mathcal{K}} m)_{|\mathcal{K}y\mathcal{K}}.$$

In particular one obtains an induced action of $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ on $\mathrm{St}_{\mathcal{K}}$.

Proof: To prove the lemma, it suffices to prove for each $\tilde{e} = \overrightarrow{\tilde{v}\tilde{v}'} \in \mathcal{T}^{o,\mathrm{st}}_{\mathcal{K},1}$ and each j that

$$\partial_{\mathcal{K}} \sum \Big\{[(e, gy'_j\mathcal{K})] : (e, gy'_j\mathcal{K}) \in \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}, (e, g\mathcal{K}) \in \mathrm{crs}(\tilde{e})\Big\} \\ = [(v', gy'_j\mathcal{K})] - [(v, gy'_j\mathcal{K})], \tag{38}$$

where $\tilde{v} = (v, g\mathcal{K})$ and $\tilde{v}' = (v', g\mathcal{K})$, and where we require that the coset representative g is fixed for all symbols.

If both, $\tilde{v}$ and $\tilde{v}'$, are stable, then $\mathrm{crs}(\tilde{e}) = \{\tilde{e}\}$ and the assertion is obvious. Let us now assume that $\tilde{v}'$ is stable and $\tilde{v}$ isn't. Then the elements in $\mathfrak{l} := \{(e, gy'_j\mathcal{K}) : (e, g\mathcal{K}) \in \mathrm{crs}(\tilde{e})\}$ are the translate of the union of $\tilde{e}$ with the edges on the half line $\underline{s}$ from $\tilde{v}$ to $\mathbf{b}(\tilde{v})$. Let $\tilde{v}_0 := (v, gy'_j\mathcal{K})$ be the initial point of $\mathfrak{l}$ and define $\tilde{w}_0, \tilde{v}_1, \tilde{w}_1, \ldots, \tilde{v}_m, \tilde{w}_m$ as the unstable vertices of $\underline{s}$ such that the segment between $\tilde{v}_i$ and $\tilde{w}_i$ consists entirely of stable edges and such that the union of these segments is the set of all stable edges on $\mathfrak{l}$. If the edge adjacent to $\tilde{v}_0$ is unstable, then we have $\tilde{w}_0 = \tilde{v}_0$. The vertex $\tilde{v}_0$ may or not be stable.

If one applies $\partial_{\mathcal{K}}$ to the sum of the edges of $\mathfrak{l}$ which are in the segment between $\tilde{v}_i$ and $\tilde{w}_i$ for $i > 0$, then the result is zero because the sum is telescoping. For $i = 0$, the sum equals $[\tilde{v}_0]$. Because $\tilde{v}$ is stable, the component of $\tilde{v}_{|\mathcal{K}y\mathcal{K}}$ for given j is precisely $[(v, gy'_j\mathcal{K})]$, i.e. $[\tilde{v}_0]$. Thus the assertion is proved in this case as well. Finally, the case that both ends are unstable is similar to the previous one and left to the reader. ∎

For any $\mathbb{Z}[\mathrm{GL}_2(K)]$-module V, we define an action of $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ on $\mathrm{St}_{\mathcal{K}} \otimes_{\mathrm{GL}_2(K)} V$ by having $y \in \mathcal{Y}$ act as $_{|\mathcal{K}y\mathcal{K}} \otimes \mathrm{id}$.

Proposition 6.22 *For any K-algebra R, the isomorphism $\Phi : \tilde{\mathbf{C}}^{\mathrm{har}}_n(\mathcal{K}, R) \to \mathrm{St}_{\mathcal{K}} \otimes_{\mathrm{GL}_2(K)} V_n(R)$ of Proposition 5.50 is Hecke-equivariant.*

We consider the complex

$$\mathbb{Z}[\mathcal{T}_{\mathcal{K},\bullet}] : \ldots \longrightarrow 0 \longrightarrow \mathbb{Z}[\overline{\mathcal{T}}^{o}_{\mathcal{K},1}] \xrightarrow{\partial} \mathbb{Z}[\mathcal{T}_{\mathcal{K},0}] \longrightarrow 0 \longrightarrow \ldots,$$

where the symbol of a simplex $(t, g\mathcal{K})$ is denoted by $\{(t, g\mathcal{K})\}$. Clearly there is a commutative diagram

$$\begin{CD}
\mathbb{Z}[\overline{\mathcal{T}}^{o}_{\mathcal{K},1}] @>{\partial}>> \mathbb{Z}[\mathcal{T}_{\mathcal{K},0}] \\
@VVV @VVV \\
\mathbb{Z}[\overline{\mathcal{T}}^{o,\mathrm{st}}_{\mathcal{K},1}] @>{\partial_{\mathcal{K}}}>> \mathbb{Z}[\mathcal{T}^{\mathrm{st}}_{\mathcal{K},0}],
\end{CD}$$

where the vertical maps are given by mapping a symbol $\{(t, g\mathcal{K})\}$ to the symbol $[(t, g\mathcal{K})]$. This map is denoted by $[\_\_]$.

We leave the proof of the following simple lemma, which uses Proposition 5.50 for the second assertion, to the reader:

Lemma 6.23 *The map which sends a harmonic cochain* $\tilde{\mathbf{c}}$ *to*

$$\sum_{(e,g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \{(e, g\mathcal{K})\} \otimes \tilde{\mathbf{c}}(e, g\mathcal{K}) \in \mathbb{Z}[\overline{\mathcal{T}}^{o}_{\mathcal{K},1}] \otimes_{\mathrm{GL}_2(K)} V_n(R)$$

defines an isomorphism $\tilde{\mathbf{C}}^{\mathrm{har}}_n(\mathcal{K}, R) \to H_1(\mathbb{Z}[\mathcal{T}_{\mathcal{K},\bullet}] \otimes_{\mathrm{GL}_2(K)} V_n(R))$. *Furthermore, the map* $[\_\_]$ *induces an isomorphism*

$$H_1(\mathbb{Z}[\mathcal{T}_{\mathcal{K},\bullet}] \otimes_{\mathrm{GL}_2(K)} V_n(R)) \longrightarrow \mathrm{St}_{\mathcal{K}} \otimes V_n(R),$$

such that the composite map is the map given in Proposition 5.50.

We define a Hecke action on $\mathbb{Z}[\mathcal{T}_{\mathcal{K},\bullet}]$, by having $\mathcal{K}y\mathcal{K}$ map the symbol $\{(t, g\mathcal{K})\}$ to $\sum_j \{(t, gy'_j\mathcal{K})\}$. The following is the first stage in the proof of Proposition 6.22.

Lemma 6.24 *If the action of* $\mathcal{K}y\mathcal{K}$ *on* $\mathbb{Z}[\mathcal{T}_{\mathcal{K},\bullet}] \otimes_{\mathrm{GL}_2(K)} V_n(R))$ *is defined as* ${}_{|\mathcal{K}y\mathcal{K}} \otimes \mathrm{id}$, *then the map* $\tilde{\mathbf{C}}^{\mathrm{har}}_n(\mathcal{K}, R) \to H_1(\mathbb{Z}[\mathcal{T}_{\mathcal{K},\bullet}] \otimes_{\mathrm{GL}_2(K)} V_n(R))$ *of the previous lemma is a Hecke equivariant isomorphism.*

PROOF: The lemma is a consequence of the following calculation:

$$\begin{aligned}
&\sum_{(e,g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \{(e, g\mathcal{K})\} \otimes (\tilde{\mathbf{c}}_{|\mathcal{K}y\mathcal{K}})(e, g\mathcal{K}) \\
&= \sum_{(e,g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \sum_j \{(e, g\mathcal{K})\} \otimes \tilde{\mathbf{c}}(e, gy_j^{-1}\mathcal{K}) \\
&\overset{\coprod y_j^{-1}\mathcal{K} = \mathcal{K}y^{-1}\mathcal{K}}{=} \sum_{(e,g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \sum_{h\mathcal{K}\subset g\mathcal{K}y^{-1}\mathcal{K}} \{(e, g\mathcal{K})\} \otimes \tilde{\mathbf{c}}(e, h\mathcal{K}) \\
&\overset{h\in g\mathcal{K}y^{-1}\mathcal{K} \Leftrightarrow g\in h\mathcal{K}y\mathcal{K}}{=} \sum_{(e,g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \sum_{g\mathcal{K}\subset h\mathcal{K}y\mathcal{K}} \{(e, g\mathcal{K})\} \otimes \tilde{\mathbf{c}}(e, h\mathcal{K}) \\
&= \sum_{(e,h\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \sum_j \{(e, hy'_j\mathcal{K})\} \otimes \tilde{\mathbf{c}}(e, h\mathcal{K}) \\
&= \sum_{(e,h\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \{(e, h\mathcal{K})\}_{|\mathcal{K}y\mathcal{K}} \otimes \tilde{\mathbf{c}}(e, h\mathcal{K}). \quad \blacksquare
\end{aligned}$$

PROOF of Proposition 6.22: We may apply the above lemma to the first isomorphism given in Lemma 6.23. Therefore to prove the proposition, it remains to show that the map in Lemma 6.23 induced from $[\_\_]$ is Hecke-equivariant, i.e., we have to show that $_{|\mathcal{K}y\mathcal{K}} \circ [\_\_] = [\_\_] \circ {}_{|\mathcal{K}y\mathcal{K}}$. We have on the one hand

$$\begin{aligned}
&[__] \circ {}_{|\mathcal{K}y\mathcal{K}}\Big(\sum_{(e,g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \{(e,g\mathcal{K})\} \otimes \tilde{\mathbf{c}}(e,g\mathcal{K})\Big) \\
&= [__]\Big(\sum_{(e,g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \sum_j \{(e,gy'_j\mathcal{K})\} \otimes \tilde{\mathbf{c}}(e,g\mathcal{K})\Big) \\
&= \sum_{(e,g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \sum_{\{j:(e,gy'_j\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}\}} \\
&\qquad \sum_{(e,g\mathcal{K})\in \mathrm{crs}(e',g\mathcal{K})} [(e,gy'_j\mathcal{K})] \otimes \tilde{\mathbf{c}}(e',g\mathcal{K})
\end{aligned}$$

and on the other

$$\begin{aligned}
&{}_{|\mathcal{K}y\mathcal{K}} \circ [__]\Big(\sum_{(e',g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \{(e',g\mathcal{K})\} \otimes \tilde{\mathbf{c}}(e',g\mathcal{K})\Big) \\
&= \sum_{(e',g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}/\{\pm 1\}} [(e',g\mathcal{K})]_{|\mathcal{K}y\mathcal{K}} \otimes \tilde{\mathbf{c}}(e',g\mathcal{K}) \\
&= \sum_{(e',g\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^o_{\mathcal{K},1}/\{\pm 1\}} \sum_{(e,g\mathcal{K})\in \mathrm{crs}((e',g\mathcal{K}))} \\
&\qquad \sum_{\{j:(e,gy'_j\mathcal{K})\in \mathrm{GL}_2(K)\backslash \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}\}} [(e,gy'_j\mathcal{K})] \otimes \tilde{\mathbf{c}}(e',g\mathcal{K}).
\end{aligned}$$

This concludes the proof of Proposition 6.23. ∎

We also record the following compatibility, which is a consequence of Propositions 6.15 and 6.22, and Theorem 5.51:

Proposition 6.25 *The isomorphism* $\mathbf{S}^2_n(\mathcal{K}) \cong \mathbf{C}^{\mathrm{St},2}_n(\mathcal{K},\mathbb{C}_\infty)$ *given by* $\Phi_{\mathcal{K}}\circ \mathrm{Res}_{\mathcal{K}}$ *is Hecke-equivariant.*

Finally, we state the following result on integral harmonic cochains, which will eventually be proved in Subsection 13.2 as a corollary to our Eichler-Shimura isomorphisms, Theorems 10.3 and 12.3, and their compatibility with the Hecke-operation, Theorem 13.2.

Proposition 6.26 *The submodules*

$$\mathbf{C}^{\mathrm{St},2}_n(\mathcal{K},A) \subset \mathbf{C}^{\mathrm{St}}_n(\mathcal{K},A)$$

of $\mathbf{C}^{\mathrm{St}}_n(\mathcal{K},K)$ *are stable under the action of* $\mathcal{H}(\mathcal{K},\mathcal{Y})$.

7 Crystals over function fields

Throughout this section, we fix a morphism $f: Y \to X$ of noetherian schemes X, Y over $\operatorname{Spec} k$ and an open immersion $j: U \to X$ with closed complement $i: Z \to X$ and ideal sheaf $\mathcal{I}$ of Z. The aim is to recall the relevant notions and facts from [4]. For a more detailed account of the theory of τ-sheaves and crystals, we refer to loc. cit. We also fix a k-algebra B essentially of finite type over k. Also recall that σ is the Frobenius on X, Y, ..., relative to k.

7.1 Basic definitions

Definition 7.1 *A* τ-sheaf over B *on a scheme* X *is a pair* $\underline{\mathcal{F}} := (\mathcal{F}, \tau_{\mathcal{F}})$ *consisting of a quasi-coherent sheaf* $\mathcal{F}$ *on* $X \times \operatorname{Spec} B$ *and an* $\mathcal{O}_{X\times \operatorname{Spec} B}$*-linear homomorphism*

$$(\sigma \times \mathrm{id})^*\mathcal{F} \xrightarrow{\tau} \mathcal{F}.$$

The τ*-sheaf* $\underline{\mathcal{F}}$ *is called* coherent *if* $\mathcal{F}$ *has this property.*

We often simply speak of τ*-sheaves on* X. The sheaf underlying a τ-sheaf $\underline{\mathcal{F}}$ will always be denoted $\mathcal{F}$. When the need arises to indicate on which sheaf τ acts, we write $\tau = \tau_{\mathcal{F}}$.

On any affine chart $\operatorname{Spec} R \subset X$ a τ-sheaf over B corresponds to a finitely generated $R \otimes B$-module M together with a $\sigma \otimes \mathrm{id}$-linear homomorphism $\tau: M \to M$. We will occasionally use the notation (M, τ) and call it a τ-module.

By $\mathbf{QCoh}_\tau(X, B)$ we denote the category whose objects are the τ-sheaves on X over B, and whose morphisms are those sheaf homomorphisms which are compatible with τ. The full subcategory of coherent τ-sheaves is denoted $\mathbf{Coh}_\tau(X, B)$. With the obvious definitions of kernel, cokernel, image and coimage, $\mathbf{Coh}_\tau(X, B)$ and $\mathbf{QCoh}_\tau(X, B)$ are abelian B-linear categories.

For a τ-sheaf $\underline{\mathcal{F}}$, we define the iterates τ^n of τ by setting inductively $\tau^0 := \mathrm{id}$ and $\tau^{n+1} := \tau \circ (\sigma \times \mathrm{id})^*\tau^n : (\sigma^{n+1} \times \mathrm{id})^*\mathcal{F} \longrightarrow \mathcal{F}$.

Definition 7.2 *A* τ*-sheaf* $\underline{\mathcal{F}}$ *is called* nilpotent *if and only if* $\tau^n_{\mathcal{F}}$ *vanishes for some* $n > 0$.

A morphism of coherent τ*-sheaves is called a* nil-isomorphism *if and only if both its kernel and cokernel are nilpotent.*

Proposition 7.3 *A homomorphism of coherent* τ*-sheaves* $\varphi: \underline{\mathcal{F}} \to \underline{\mathcal{G}}$ *is a nil-isomorphism if and only if there exist* $n \geq 0$ *and a morphism of* τ*-sheaves* α *making the following diagram commute:*

$$\begin{tikzcd}
(\sigma^n \times \mathrm{id})^*\underline{\mathcal{F}} \arrow[r, "\tau^n"] \arrow[d, "(\sigma^n\times\mathrm{id})^*\varphi"'] & \underline{\mathcal{F}} \arrow[d, "\varphi"] \\
(\sigma^n \times \mathrm{id})^*\underline{\mathcal{G}} \arrow[r, "\tau^n"] \arrow[ur, "\alpha"] & \underline{\mathcal{G}}.
\end{tikzcd}$$

It is shown in [4], Chap. 2, that the nil-isomorphisms in $\mathbf{Coh}_\tau(X, B)$ form a saturated multiplicative system, denoted by $\mathcal{S}$. One can thus make the following definition.

Definition 7.4 *The category* $\mathbf{Crys}(X, B)$ *of* B-crystals on X *is the localization of* $\mathbf{Coh}_\tau(X, B)$ *with respect* $\mathcal{S}$.

Thus the category of crystals is obtained from that of coherent τ-sheaves by 'formally inverting all nil-isomorphisms'.

If the need arises to distinguish different kinds of morphisms, we use the following convention. Dotted arrows $\dashrightarrow$ indicate homomorphisms in $\mathbf{Crys}(X,B)$, solid arrows $\longrightarrow$ denote homomorphisms in $\mathbf{QCoh}_\tau(X,B)$ and double arrows $\Longrightarrow$ denote nil-isomorphisms in $\mathbf{QCoh}_\tau(X,B)$. Mostly however, we use $\longrightarrow$ in all three cases. In particular, any morphism $\underline{\mathcal{F}} \dashrightarrow \underline{\mathcal{G}}$ in $\mathbf{Crys}(X,B)$ is represented by a diagram $\underline{\mathcal{F}} \Longleftarrow \underline{\mathcal{H}} \to \underline{\mathcal{G}}$ in $\mathbf{Coh}_\tau(X,B)$.

For a τ-sheaf $\underline{\mathcal{F}}$ on X over B, we define $\underline{\mathcal{F}}^\tau := \Gamma(X \times \operatorname{Spec} A, \underline{\mathcal{F}})^\tau$ as the B-module of τ-invariant global sections of $\underline{\mathcal{F}}$. The following result is an immediate consequence of the above definitions.

Proposition 7.5 *The functor $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^\tau$ from $\mathbf{Coh}_\tau(X,B)$ to B-modules is invariant under nil-isomorphisms and passes therefore to a functor on $\mathbf{Crys}(X,B)$, which is again denoted $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^\tau$.*

Proposition 7.3 provides us with a standard presentation for morphisms of B-crystals:

Proposition 7.6 *Any morphism $\varphi\colon \underline{\mathcal{F}} \dashrightarrow \underline{\mathcal{G}}$ in $\mathbf{Crys}(X,B)$ can be represented for suitable n by a diagram*

$$\underline{\mathcal{F}} \overset{\tau^n}{\Longleftarrow} (\sigma^n \times \mathrm{id})^* \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{G}}.$$

We call a τ-sheaf $\underline{\mathcal{F}}$ *(locally) free*, if its underlying sheaf $\mathcal{F}$ is (locally) free. We call a crystal *(locally) free*, if it may be represented by a (locally) free τ-sheaf.

We now give some examples which are explained in greater detail in [3]:

Example 7.7 (a) An *A-motive on X of rank r* is a pair $(\underline{\mathcal{M}}, \mathrm{char}_{\underline{\mathcal{M}}})$ where $\underline{\mathcal{M}} \in \mathbf{Coh}_\tau(X,A)$ is locally free of rank r and $\mathrm{char}_{\underline{\mathcal{M}}} : X \to \operatorname{Spec} A$ is a morphism of schemes such that the following conditions hold:

(i) The sheaf $\operatorname{Coker}((\sigma \times \mathrm{id})^* \mathcal{M} \xrightarrow{\tau} \mathcal{M})$ vanishes on the complement of the graph of $\mathrm{char}_{\underline{\mathcal{M}}}$ inside $X \times \operatorname{Spec} A$.

(ii) For every geometric point $i_{\bar{x}} : \bar{x} \hookrightarrow X$, so that $\bar{x}$ is the spectrum of an algebraically closed field, the τ-sheaf $i_{\bar{x}}^* \underline{\mathcal{M}}$ is a Drinfeld-Anderson A-motive of rank r in the sense of [41], Def. 5.1.

The pair $(\underline{\mathcal{M}}, \mathrm{char}_{\underline{\mathcal{M}}})$ is also referred to as a *family of A-motives on X of rank r*, and $\mathrm{char}_{\underline{\mathcal{M}}}$ is called the *characteristic of $\underline{\mathcal{M}}$*.

(b) Let $\underline{\varphi} := (\varphi, \mathcal{L})$ be a Drinfeld-A-module of rank r on an A-scheme X. The pair $(\varphi, \mathcal{L})$ defines an A-motive on X of rank r in the following way. Denote by $\tau' \in \operatorname{End}_{\mathfrak{G}/X}(\mathbb{G}_{a,X})$ the Frobenius on $\mathbb{G}_{a,X}$ relative to k. Define

$$\mathcal{M}(\varphi) := \operatorname{Hom}_{\mathfrak{G}/X}(\mathcal{L}, \mathbb{G}_{a,X}).$$

This is naturally a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The action of $a \in A$ is defined as right composition with $\varphi(a)$, and the action of τ as left composition with τ'. This defines an $\mathcal{O}_X \otimes A$-linear map $\tau\colon \mathcal{M}(\varphi) \to (\sigma \times \mathrm{id})_* \mathcal{M}(\varphi)$, i.e., it makes $(\mathcal{M}(\varphi), \tau)$ into a τ-sheaf $\underline{\mathcal{M}}(\varphi)$. The sheaf $\mathcal{M}(\varphi)$ is in fact locally free of rank r on $X \times \operatorname{Spec} A$, cf. [11].

The characteristic of φ, or of $\underline{\mathcal{M}}(\varphi)$, is the structure morphism $\mathrm{char}_\varphi\colon X \to \operatorname{Spec} A$ which makes X into an A-scheme. Equivalently, char_φ is the morphism of schemes corresponding to the ring homomorphism

$$A \to \operatorname{End}_{\mathcal{O}_X}(\operatorname{Lie}(\mathcal{L})) \cong \Gamma(X, \mathcal{O}_X)$$

induced by φ, where $\mathrm{Lie}(\mathcal{L})$ is the tangent space to $\mathcal{L}$ along the zero section and canonically isomorphic to $\mathcal{L}$. The pair $(\underline{\mathcal{M}}(\varphi), \mathrm{char}_\varphi)$ is an A-motive of rank r. The verification of condition (ii) is given in [1], (0.2), (0.3), (0.4). The verification of (i) is an easy consequence.

Further examples are provided in the following section by a) the τ-sheaf attached to a pure Drinfeld-Anderson motive, cf. Definition 9.1, and b) the A-motive associated to an A-module, cf. 9.9.

Definition 7.8 *For an admissible $\mathcal{K}$, we define the τ-sheaf attached to the universal Drinfeld-module $\varphi_{\mathcal{K}}$ as $\underline{\mathcal{F}}_{\mathcal{K}} := \mathcal{M}(\underline{\varphi}_{\mathcal{K}})$.*

7.2 Functors

This subsection is somewhat informal and mainly serves to fix some notation and recall some results from [4], § 2, 3. For a survey in the case where B is regular, one may also consult [3], § 1. Note also that in Section 8 some of the quoted results are presented in greater detail in an analytic context.

On the category of quasi- and coherent τ-sheaves, one can define a functor f^* for any f, and functors $R^i f_*$ for proper f and $i \geq 0$. On the underlying sheaves these functors are defined in the usual way, as for instance in Hartshorne, [26]. In both cases, the functoriality of f^* and $R^i f_*$ yields a canonical choice for τ. (To define $R^i f_*$ on the category of quasi-coherent τ-sheaf the properness hypothesis is unnecessary.) In [4], all these functors are developed in a suitable derived context and it turns out a posteriori that on τ-sheaves they yield the functors $R^i f_*$ defined above.

Proposition 7.9 *If f is proper, the functor f^* is left adjoint to f_*. In particular, one has an adjunction morphism*

$$\mathrm{id} \xrightarrow{\mathrm{adj}} f_* f^*.$$

If f is finite flat, then there is a trace map $f_* f^* \xrightarrow{\mathrm{trace}} \mathrm{id}$, which is defined locally for $Y = \operatorname{Spec} S$, $X = \operatorname{Spec} R$ and a τ-module M as

$$\mathrm{trace}_{S/R} \otimes \mathrm{id}_M : S \otimes_R M \to M$$

with the induced τ, and where $\mathrm{trace}_{S/R}$ is the usual trace map for the finite flat ring S over R. Because $\mathrm{trace}_{S/R}$ is zero for inseparable morphism $R \to S$, one has the following lemma, which we state for later use.

Lemma 7.10 *If f is inseparable and finite flat, then* $\mathrm{trace}\colon f_* f^* \to \mathrm{id}$ *is the zero map.*

One also defines the bi-functor $\otimes$, which gives the tensor product of τ-sheaves, and $\otimes_B B'$ for any morphism $h\colon B \to B'$, which may be regarded as change of coefficients. These functors have higher left derived functors. We single out a certain class of acyclic objects.

Definition 7.11 *A τ-sheaf $\underline{\mathcal{F}}$ is called* of pullback type *if there exists a coherent sheaf $\mathcal{F}_0$ on X such that $\mathcal{F} \cong \mathrm{pr}_1^* \mathcal{F}_0$.*

A crystal is called of pullback type*, if it can be represented by a τ-sheaf of pullback type.*

In particular, if X is affine, then any locally free crystal is of pullback type.

Lemma 7.12 *Crystals and τ-sheaves of pullback type are acyclic for the functors $\otimes$ and $\otimes_B B'$.*

In [4], the notion of *flat crystals* is developed. It gives a general framework for acyclic objects for $\otimes$ and $\otimes_B B'$. For simplicity, and because this will suffice for most of our purposes, we do not introduce the more general notion.

The one-object for $\otimes$ is denoted $\underline{\mathbb{1}}_{X,B}$. Its underlying sheaf is $\mathcal{O}_{X\times \operatorname{Spec} B}$, the morphism

$$\tau\colon (\sigma\times \mathrm{id})^*\mathcal{O}_{X\times \operatorname{Spec} B} \longrightarrow \mathcal{O}_{X\times \operatorname{Spec} B}$$

is the isomorphism underlying $\sigma \times \mathrm{id}$. The object in $\mathbf{Crys}(X,B)$ represented by $\underline{\mathbb{1}}_{X,B}$ is called the *unit crystal on X over B*.

For a coherent τ-sheaf $\underline{\mathcal{F}}$, we also define $\operatorname{Sym}^n \underline{\mathcal{F}}$ and $\bigwedge^n \underline{\mathcal{F}}$, the n-th symmetric and exterior powers of $\underline{\mathcal{F}}$. The construction on the underlying sheaf is as in [26], Ex. II.5.16, and again there is a canonical choice for τ. Note that both $\operatorname{Sym}^n \underline{\mathcal{F}}$ and $\bigwedge^n \underline{\mathcal{F}}$ are defined as quotients of $\underline{\mathcal{F}}^{\otimes n}$.

Our main examples of τ-sheaves will be families of A-motives, so it is natural to ask when tensor products, symmetric and exterior powers are again families of A-motives. The key to this is purity, and the precise results will be given in Subsection 9.1.

It is not difficult to see that the functors f^*, $R^i f_*$, $\otimes$, Sym^n, $\bigwedge^n$ and $\otimes_B B'$ are compatible with localization at nil-isomorphisms and hence they induce functors on the category of crystals.

The following theorem summarizes some results that hold in $\mathbf{Crys}(X,B)$ but not necessarily in $\mathbf{Coh}_\tau(X,B)$.

Theorem 7.13 *(i) The functor $f^*\colon \mathbf{Crys}(X,B)\to \mathbf{Crys}(Y,B)$ is exact.*

(ii) A crystal on X is zero if all its stalks are zero. If X is of finite type over a field, then it suffices that all the stalks at closed points of X are zero.

(iii) There exists a functor $j_!\colon \mathbf{Crys}(U,B)\to\mathbf{Crys}(X,B)$, called extension by zero*, which is uniquely characterized by the properties $j^*j_! = \mathrm{id}_{\mathbf{Crys}(U,B)}$ and $i^*j_! = 0$.*

(iv) If $\underline{\mathcal{F}} \in \mathbf{Crys}(U,B)$ is of pullback type, then so is $j_!\underline{\mathcal{F}}$. Also if $\underline{\mathcal{G}} \in \mathbf{Crys}(Y,B)$ is of pullback type, then so are the $R^i f_*\underline{\mathcal{G}}$ for proper f.

The characterization of $j_!$ directly implies that a sufficient condition for a given τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on X to represent $j_!\underline{\mathcal{F}}$ for some τ-sheaf $\underline{\mathcal{F}}$ on U, is that $j^*\underline{\tilde{\mathcal{F}}} = \underline{\mathcal{F}}$ and $i^*\underline{\tilde{\mathcal{F}}}$ is nilpotent. This indicates how to construct $j_!$ for $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U,B)$ regarded as a crystal:

Take any coherent sheaf $\tilde{\mathcal{F}}$ on $X\times\operatorname{Spec} B$, whose restriction to $U\times \operatorname{Spec} B$ agrees with $\mathcal{F}$. One verifies that for $n \gg 0$, the morphism $\tau_{\mathcal{F}}$ extends to a morphism

$$\tau\colon (\sigma\times\mathrm{id})^*\mathcal{I}^n\tilde{\mathcal{F}} \longrightarrow \mathcal{I}^n\tilde{\mathcal{F}}$$

such that the restriction of τ to $i^*\mathcal{I}^n\tilde{\mathcal{F}}$ is nilpotent. The resulting τ-sheaf will then represent $j_!\underline{\mathcal{F}}$.

Definition 7.14 (Cohomology with compact supports) *Say f is compactifiable, i.e., $f = \bar{f}j$ for some $\bar{f}\colon \bar{Y}\to X$ which is proper and some $j\colon Y\to\bar{Y}$ which is an open immersion. Then one defines*

$$R^i f_! := R^i\bar{f}_* \circ j_!\colon \mathbf{Crys}(Y,B) \longrightarrow \mathbf{Crys}(X,B).$$

Standard arguments show that the definition is independent of the chosen factorization, e.g. [38], Ch. VI, §3. Furthermore, due to a result of Nagata, any morphism $f : Y \to X$ between schemes of finite type over k is compactifiable, and so in this situation the $R^i f_!$ exist, cf. [37]. The previous theorem also implies:

Proposition 7.15 *If $\underline{\mathcal{F}} \in \mathbf{Crys}(Y, B)$ is of pullback type, then so are the $R^i f_! \underline{\mathcal{F}}$ for any morphism f.*

7.3 Relations with the étale site

For this subsection we assume that B is a finite k-algebra. Let $\mathbf{\acute{E}t}(X, B)$ be the category of étale sheaves of B-modules and $\mathbf{\acute{E}t}_c(X, B)$ its full subcategory of constructible sheaves.

Let $\mathrm{pr}_1 : X \times \operatorname{Spec} B \to X$ be the projection onto the first factor. For a coherent τ-sheaf $\underline{\mathcal{F}}$, the sheaf $\mathrm{pr}_{1*} \mathcal{F}$ is quasi-coherent on X, and we denote by $\underset{\sim}{\mathcal{F}}$ the associated étale sheaf, cf. [38], p. 48. There is a canonical isomorphism $\underset{\sim}{(\sigma \times \mathrm{id})_* \mathcal{F}} \to \underset{\sim}{\mathcal{F}}$ of étale sheaves of B-modules, because σ is the identity on the topological space underlying X. By composing this isomorphism with the map on the étale site induced from the adjoint $\tau^{\#} : \underline{\mathcal{F}} \to (\sigma \times \mathrm{id})_* \underline{\mathcal{F}}$ of τ, one obtains an endomorphism $\tau_{\text{ét}} : \underset{\sim}{\mathcal{F}} \to \underset{\sim}{\mathcal{F}}$ on the étale site.

Definition 7.16 *We define $\underline{\mathcal{F}}_{\text{ét}} := \mathrm{Ker}(1 - \tau_{\text{ét}} : \underset{\sim}{\mathcal{F}} \to \underset{\sim}{\mathcal{F}})$. This assignment yields a functor*

$$\mathbf{Coh}_\tau(X, B) \to \mathbf{\acute{E}t}(X, B) : \underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}.$$

Generalizing Artin-Schreier theory, the following is proved in [30], Thm. 4.1.1:

Theorem 7.17 *The functor $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}$ defines an equivalence between the categories $\{\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, B) : \tau_{\mathcal{F}}$ is an isomorphism$\}$ and $\{\mathsf{F} \in \mathbf{\acute{E}t}_c(X, B) :$ F is locally constant$\}$ and there is an explicit functor $P_\tau^{\text{ét}}$ which defines an inverse.*

In [4], § 8, this is further elaborated. The following theorem summarizes the main results:

Theorem 7.18 *The functor $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}$ maps nil-isomorphisms to isomorphisms, i.e., it factors via $\mathbf{Crys}(X, B)$. Its image lies in $\mathbf{\acute{E}t}_c(X, B)$. The induced functor*

$$\epsilon \colon \mathbf{Crys}(X, B) \to \mathbf{\acute{E}t}_c(X, B)$$

is an equivalence of categories. Finally the functor ϵ is compatible with any of the functors f^*, $\otimes$, $\otimes_B B'$, Sym^n, $\bigwedge^n$ and $R^i f_!$.

In analogy to the unit crystal, we define $\mathbb{1}_{X,B}^{\text{ét}}$ as the constant étale sheaf on X with B-coefficients.

The main example to keep in mind is the following: Let $\varphi \colon A \to R\{\tau\}$ be a Drinfeld module of rank r, where R is a normal noetherian ring over k, and fix a non-zero proper ideal $\mathfrak{n}$ of A. Define $\varphi[\mathfrak{n}](S)$ for any morphism $u \colon R \to S$ by

$$\varphi[\mathfrak{n}](S) := \{f \in S | \forall a \in \mathfrak{n}\, \varphi_a : f = 0\}.$$

One easily checks that $\varphi[\mathfrak{n}]$ defines a sheaf on the big étale site of affines over $\operatorname{Spec} R$, the sheaf of $\mathfrak{n}$-torsion points of φ. Note that the functor $\varphi[\mathfrak{n}]$ is compatible with direct limits. Finally define (M, τ) as the τ-module whose associated τ-sheaf is $\underline{\mathcal{F}} := \underline{\mathcal{M}}(\varphi)$. The following result is modeled after, [1], Prop. 1.8.3, where Anderson proves it in the case where R is a separably closed field.

Proposition 7.19 *Suppose the characteristic of φ is disjoint from $\mathfrak{n}$. Then there is an isomorphism of étale sheaves*

$$\varphi[\mathfrak{n}] \cong \mathsf{H}\mathscr{om}_A((\underline{\mathcal{F}}/\mathfrak{n}\underline{\mathcal{F}})_{\text{ét}}, \mathfrak{n}^{-1}\Omega_A/\Omega_A)$$

on the small étale site over $\operatorname{Spec} R$*, where* $\mathfrak{n}^{-1}\Omega_A/\Omega_A$ *is regarded as the constant sheaf on* $\operatorname{Spec} R$*.*

PROOF: Let $u\colon R \to S$ be étale. Following Anderson, we define on $\operatorname{Hom}_k(A/\mathfrak{n}, S)$ the structure of left $S\{\tau\} \otimes A$-module by

$$\begin{aligned}(s \cdot h)(\bar{a}) &= s(h(\bar{a})),\\ (a \cdot h)(\bar{a}) &= h(a\bar{a}),\\ (\tau h)(\bar{a}) &= (h(\bar{a}))^q\end{aligned}$$

for $\bar{a} \in A/\mathfrak{n}$, $h \in \operatorname{Hom}_k(A/\mathfrak{n}, S)$, $s \in S$ and $a \in A$. With this definition, the map

$$\varphi[\mathfrak{n}](S) \longrightarrow \operatorname{Hom}_{S\{\tau\}\otimes A}(u^*\underline{\mathcal{F}}/\mathfrak{n}u^*\underline{\mathcal{F}}, \operatorname{Hom}_k(A/\mathfrak{n}, S))$$
$$e \mapsto (\bar{m} \mapsto (\bar{a} \mapsto \bar{m}(\bar{a}e)))$$

for $e \in \varphi[\mathfrak{n}](S)$, $\bar{m} \in u^*\underline{\mathcal{F}}/\mathfrak{n}u^*\underline{\mathcal{F}}$ and $\bar{a} \in A/\mathfrak{n}$ is well-defined and functorial in u. Via the residue map $\operatorname{Res}_\infty$ at ∞, one has a natural isomorphism $\operatorname{Hom}_k(A/\mathfrak{n}, S) \cong \mathfrak{n}^{-1}\Omega_A/\Omega_A \otimes_k S$. The inclusion

$$(u^*\underline{\mathcal{F}}/\mathfrak{n}u^*\underline{\mathcal{F}})^\tau \otimes_k S \hookrightarrow u^*\underline{\mathcal{F}}/\mathfrak{n}u^*\underline{\mathcal{F}},$$

which is again functorial in s, induces therefore a map of étale sheaves

$$\varphi[\mathfrak{n}] \longrightarrow \mathsf{H}\mathscr{om}_A((\underline{\mathcal{F}}/\mathfrak{n}\underline{\mathcal{F}})_{\text{ét}}, \mathfrak{n}^{-1}\Omega_A/\Omega_A) =: \mathsf{F}_\mathfrak{n}. \tag{39}$$

By the remark preceding the proposition, the latter map is an isomorphism whenever S is a separably closed field. Because $\mathfrak{n}$ is disjoint from the characteristic of φ, both $\varphi[\mathfrak{n}]$ and $\mathsf{F}_\mathfrak{n}$ are locally free of rank r over $A/\mathfrak{n}$. Hence the above map, being an isomorphism for algebraically closed fields, is an isomorphism on the étale site above $\operatorname{Spec} R$. ∎

The above proposition and the compatibility of ϵ with Sym^n applied to the universal Drinfeld module $\varphi_\mathcal{K}$ immediately yield the following:

Corollary 7.20 *Let $\mathcal{K}$ be admissible and $\mathfrak{n}$ a non-zero ideal of A which is relatively prime to the minimal conductor $\mathfrak{n}'$ of $\mathcal{K}$. Then for any $k \geq 0$, on $\mathfrak{M}_\mathcal{K}$ (regarded as a scheme over* $\operatorname{Spec} A(\mathfrak{n}')$*) there is a canonical isomorphism*

$$\operatorname{Sym}^n \varphi_\mathcal{K}[\mathfrak{n}] \longrightarrow \operatorname{Sym}^n \mathsf{H}\mathscr{om}_A((\underline{\mathcal{F}_\mathcal{K}} \otimes_A A/\mathfrak{n})_{\text{ét}}, \mathfrak{n}^{-1}\Omega_A/\Omega_A).$$

Fix a place v of A. We introduce the étale realization of $\underline{\mathcal{F}_\mathcal{K}}$ as follows.

Definition 7.21 *For v prime to the minimal conductor $\mathfrak{n}'$ of $\mathcal{K}$, the constructible étale A_v-sheaf $\underline{\mathcal{F}_\mathcal{K}}^{\text{ét},v}$ on* $\operatorname{Spec} A(\mathfrak{n}')$ *is defined as the inverse limit system given by*

$$((\underline{\mathcal{F}_\mathcal{K}}/(\mathfrak{p}_v)^n\underline{\mathcal{F}_\mathcal{K}})_{\text{ét}})_{n\in\mathbb{N}}.$$

8 Analytic τ-sheaves and crystals

Throughout this section, let us fix a complete valued field $(L, |\;|_L)$, or for short simply $(L, |\;|)$, with $K_\infty \subset L \subset \mathbb{C}_\infty$. Rigid spaces are denoted by $\mathfrak{X}$, $\mathfrak{Y}$, etc. All rigid spaces will be over L and all products are formed over L. If X is a scheme over L, then by X^{rig} or X_L^{rig}, we denote the associated rigid space in the sense of [5], § 9.3.4. By B we denote a ring that is essentially of finite type over k, and by $\mathfrak{B}$ a rigid analytic Stein domain over L, cf. [32]. Define $\mathcal{B} := \Gamma(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})$. The rings B and $\mathcal{B}$ will serve as coefficient rings. The main example is the one where $\mathfrak{B} = \mathrm{Spec}(L \otimes_k B)^{\mathrm{rig}}$.

The goal is to introduce two kinds of rigid analytic sites of τ-sheaves and crystals. These will be needed in Subsection 9.2 to develop Anderson's uniformizability machine on a rigid analytic base and not just pointwise. As in Anderson, one can either have algebraic coefficients, in which case one has $B = A$, or analytic coefficients, in which case $\mathcal{B}$ is either the ring of entire functions on $\mathfrak{A} := (L \otimes A)^{\mathrm{rig}}$ or on the 'unit disc' $\mathfrak{D}_A$ of $\mathfrak{A}$, cf. Subsection 8.6.

We first introduce the rigid site with $\mathcal{B}$-coefficients and redo some of the results of the previous section for this site. In particular, we define the notions of rigid τ-sheaf and rigid crystal and we introduce various functors for these. As we want to use the results of this subsection merely as a 'tool box', we will not attempt to develop the theory to its fullest. Therefore we do not discuss it in a derived context, as was done in [4], but stay entirely within the framework of coherent (rigid analytic) sheaves. Certain notions, however, are developed with an eye toward generalizations to higher rank cases.

We then introduce a rigid site with B-coefficients. It is more of an auxiliary nature, and so we neither need, nor do bother defining cohomological functors for it. Finally, we define a functor from algebraic to rigid crystals and show that it is compatible with all the relevant functors. We conclude with some remarks on how to compute cohomology for proper rigid morphisms using Čech covers.

8.1 Basic definitions

For a rigid space $\mathfrak{X}$, we define by $\sigma_{\mathfrak{X}}$ the Frobenius which acts on affinoid rings as the q-power map. By $\sigma_{L,\mathfrak{B}}$, we denote the pullback of the Frobenius on L along $\mathrm{Spm}\,\mathfrak{B} \to \mathrm{Spm}\,L$, i.e., it acts as the q-power map on coefficients in L and as the identity on variables of $\mathcal{B}$. Finally $\tilde{\sigma}_{\mathfrak{B}}$ denotes the Frobenius of $\mathfrak{B}$ relative to L, which acts on points by mapping their coordinates to q-th powers. In particular $\sigma_{\mathfrak{B}} = \sigma_{L,\mathfrak{B}}\tilde{\sigma}_{\mathfrak{B}}$.

We define $\sigma_{\mathfrak{X}/\mathcal{B}} := \sigma_{\mathfrak{X}} \times \sigma_{L,\mathfrak{B}}$, . The morphism $\sigma_{\mathfrak{X}/\mathfrak{B}}$ will take the role played by $\sigma \times \mathrm{id}$ in the algebraic setting.

Definition 8.1 *A* rigid τ-sheaf over $\mathcal{B}$ on $\mathfrak{X}$ *is a pair* $\underline{\tilde{\mathcal{F}}} := (\tilde{\mathcal{F}}, \tau_{\tilde{\mathcal{F}}})$ *consisting of a coherent sheaf* $\tilde{\mathcal{F}}$ *of* $\mathcal{O}_{\mathfrak{X}\times\mathfrak{B}}$*-modules and an* $\mathcal{O}_{\mathfrak{X}\times\mathfrak{B}}$*-linear homomorphism*

$$(\sigma_{\mathfrak{X}} \times \sigma_{L,\mathcal{B}})^*\tilde{\mathcal{F}} \xrightarrow{\tau_{\tilde{\mathcal{F}}}} \tilde{\mathcal{F}}.$$

The corresponding category of rigid τ-sheaves over $\mathcal{B}$ is denoted $\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathfrak{B})$. The morphisms are morphisms of the underlying sheaves, i.e., of sheaves of $\mathcal{O}_{\mathfrak{X}\times\mathfrak{B}}$-modules, which are compatible with the morphism τ. Let $\mathcal{B}^\sigma$ denote the elements of $\mathcal{B}$ which are invariant under the Frobenius relative to k. Then $\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathfrak{B})$ is an abelian $\mathcal{B}^\sigma$-linear category, where kernel, cokernel, image and coimage are defined as for the underlying category of sheaves.

Let $j : \mathfrak{U} \to \mathfrak{X}$ be an affinoid subdomain of $\mathfrak{X}$. Then by the restriction $\underline{\tilde{\mathcal{F}}}_{|\mathfrak{U}}$ of a rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{X}$ to $\mathfrak{U}$, we mean the pullback under $(j \times \mathrm{id}_{\mathfrak{B}})^*$ of the underlying sheaf, together with the morphism induced by τ.

As in Section 7.1, we define the iterates τ^n and make the following definition.

Definition 8.2 *A rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ is called* nilpotent *if and only if for each affinoid subdomain $\mathfrak{U}$ of $\mathfrak{X}$, there exists an $n > 0$ such that the restriction of $\tau^n_{\tilde{\mathcal{F}}}$ to $\mathfrak{U}$ vanishes.*

A homomorphism of rigid τ-sheaves is called a nil-isomorphism *if and only if both its kernel and cokernel are nilpotent.*

Remark 8.3 An equivalent definition of nilpotence for a rigid τ-sheaf $\tilde{\mathcal{F}}$ on $\mathfrak{X}$ is that there exists an admissible affine cover $\{\mathfrak{U}_i\}$ of $\mathfrak{X}$ such that for each i there exists an $n_i \in \mathbb{N}$ such that the restriction of $\tau^{n_i}_{\tilde{\mathcal{F}}}$ to $\mathfrak{U}_i$ vanishes.

In particular, if $\mathfrak{X}$ has a finite cover by affinoids, e.g., if $\mathfrak{X}$ is proper, then $\tilde{\mathcal{F}}$ is nilpotent if and only if there exists an $n > 0$ such that $\tau^n_{\tilde{\mathcal{F}}} = 0$.

The proof of [4], Prop. 2.3.8, applies verbatim to rigid τ-sheaves and yields the following analogue of Proposition 7.3:

Proposition 8.4 *Suppose that $\mathfrak{X}$ has a finite cover by affinoids. Then a homomorphism of rigid τ-sheaves $\varphi\colon \underline{\tilde{\mathcal{F}}} \to \underline{\tilde{\mathcal{G}}}$ on $\mathfrak{X}$ over $\mathcal{B}$, is a nil-isomorphism if and only if there exist $n \geq 0$ and a homomorphism of τ-sheaves α making the following diagram commute:*

$$\begin{tikzcd}
(\sigma^n_{\mathfrak{X}/\mathcal{B}})^*\underline{\tilde{\mathcal{F}}} \arrow[r, "\tau^n"] \arrow[d, "(\sigma^n_{\mathfrak{X}/\mathfrak{B}})^*\varphi"'] & \underline{\tilde{\mathcal{F}}} \arrow[d, "\varphi"] \\
(\sigma^n_{\mathfrak{X}/\mathfrak{B}})^*\underline{\tilde{\mathcal{G}}} \arrow[r, "\tau^n"] \arrow[ur, "\alpha"] & \underline{\tilde{\mathcal{G}}}.
\end{tikzcd}$$

Following the proofs of [4], Prop 2.3.5 and Lem. 2.3.6., one shows that the nil-isomorphisms form a Serre subcategory of the category of all rigid τ-sheaves — Note that one does not need Lemma 2.2.4 of loc. cit. Hence the nil-isomorphisms in $\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},\mathcal{B})$ form a saturated multiplicative system, denoted by $\mathcal{S}_\mathcal{B}$. One can thus make the following definition.

Definition 8.5 *The category $\widetilde{\mathbf{Crys}}(\mathfrak{X},\mathcal{B})$ of* rigid $\mathcal{B}$-crystals on $\mathfrak{X}$ *is the localization of $\mathbf{Coh}_\tau(\mathfrak{X},\mathcal{B})$ with respect to $\mathcal{S}_\mathcal{B}$.*

The proof of [4], Prop. 2.4.4, can be easily adapted to rigid τ-sheaves and yields the following analogue of Proposition 7.6.

Proposition 8.6 *Assume that $\mathfrak{X}$ has a finite cover by affinoids. Then any morphism $\varphi : \underline{\tilde{\mathcal{F}}} \dashrightarrow \underline{\tilde{\mathcal{G}}}$ in $\widetilde{\mathbf{Crys}}(\mathfrak{X},\mathcal{B})$ can be represented for suitable n by a diagram*

$$\underline{\tilde{\mathcal{F}}} \overset{\tau^n}{\Longleftarrow} (\sigma^n_{\mathfrak{X}/\mathcal{B}})^*\underline{\tilde{\mathcal{F}}} \longrightarrow \underline{\tilde{\mathcal{G}}}.$$

8.2 Basic functors

Throughout this subsection we fix a morphism $f : \mathfrak{Y} \to \mathfrak{X}$ of rigid spaces. The basic functors which we will introduce in this subsection are pullback, tensor product, change of coefficients and proper pushforward. As a general rule, we do *not give* proofs, unless they differ substantially from those in [4], § 3.

Definition 8.7 (Pullback) *For a rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{X}$ over $\mathcal{B}$ denote by $f^*\underline{\tilde{\mathcal{F}}}$ the rigid τ-sheaf on $\mathfrak{Y}$ consisting of $(f\times\mathrm{id})^*\tilde{\mathcal{F}}$ and the composite homomorphism*

$$\begin{tikzcd}
\sigma^*_{\mathfrak{X}/\mathcal{B}}(f\times\mathrm{id})^*\tilde{\mathcal{F}} \arrow[r, "\tau_{f^*\tilde{\mathcal{F}}}"] \arrow[d, equal] & (f\times\mathrm{id})^*\tilde{\mathcal{F}}. \\
(f\times\mathrm{id})^*\sigma^*_{\mathfrak{X}/\mathcal{B}}\tilde{\mathcal{F}} \arrow[ur, "(f\times\mathrm{id})^*\tau_{\tilde{\mathcal{F}}}"'] &
\end{tikzcd}$$

For any homomorphism $\varphi\colon\underline{\tilde{\mathcal{F}}}\to\underline{\tilde{\mathcal{F}}}'$ we abbreviate $f^*\varphi := (f\times\mathrm{id})^*\varphi$.

This defines a $\mathcal{B}^\sigma$-linear functor $f^*:\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},\mathcal{B})\longrightarrow\widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y},\mathcal{B})$, which is clearly left exact. When f is flat, f^* is exact. In general, its exactness is governed by the associated Tor-objects $L_if^*\underline{\tilde{\mathcal{F}}}$, cf. [4], § 3.

Proposition 8.8 *(a) If $\underline{\mathcal{F}}$ is nilpotent, then so are all $L_if^*\underline{\mathcal{F}}$.*

(b) If φ is a nil-isomorphism, then so is $f^*\varphi$.

(c) The above functor on τ-sheaves induces a unique $\mathcal{B}^\sigma$-linear functor between abelian categories

$$f^*\colon\widetilde{\mathbf{Crys}}(\mathfrak{X},\mathcal{B})\longrightarrow\widetilde{\mathbf{Crys}}(\mathfrak{Y},\mathcal{B}).$$

(d) For any two morphisms $\mathfrak{Z}\xrightarrow{g}\mathfrak{Y}\xrightarrow{f}\mathfrak{X}$ there is a natural isomorphism of functors $(fg)^*\cong g^*f^*$, both on τ-sheaves and on crystals.

Our next aim is to show that, as in the algebraic setting, the functor f^* on $\mathcal{B}$-crystals is *exact.*

Theorem 8.9 *(a) For any rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{X}$ and any $i\geq 1$, the τ-sheaf $L_if^*\underline{\tilde{\mathcal{F}}}$ is nilpotent.*

(b) The functor $f^*\colon\mathbf{Crys}(\mathfrak{X},\mathcal{B})\longrightarrow\mathbf{Crys}(\mathfrak{Y},\mathcal{B})$ is exact.

Proof: The assertions of the theorem are local in $\mathfrak{X}$ and $\mathfrak{Y}$, and so we assume that $\mathfrak{Y}\to\mathfrak{X}$ arises from a map of affinoids $\tilde{f}:\mathcal{R}\to\mathcal{S}$. We write $\mathcal{M}$ for the $\mathcal{R}$-module underlying $\underline{\tilde{\mathcal{F}}}$. By the definition of affinoid, it follows that we can factor $\tilde{f}$ as

$$\mathcal{R}\xrightarrow{\tilde{f}_1}\mathcal{R}\langle\langle T_1,\ldots,T_n\rangle\rangle\overset{\tilde{f}_2}{\twoheadrightarrow}\mathcal{S}.$$

Since the first map is flat, we may assume that $\tilde{f}$ is a surjection, say with kernel I. Finally note that it suffices to prove (a), as (b) is an immediate consequence.

For (a) observe that by the usual technique of dimension shifting, it is enough to show that $\mathrm{Tor}_1^{\mathcal{R}}(\mathcal{M},\mathcal{S})$ with the induced τ is nilpotent for any rigid τ-module $(\mathcal{M},\tau)$. Because $\mathcal{M}$ is finitely generated, there exists a short exact sequence $0\to\mathcal{N}\to\mathcal{R}^n\to\mathcal{M}\to 0$ for a suitable $n\in\mathbb{N}$ and and a suitable $\mathcal{R}$-module $\mathcal{N}$. We consider the four-term exact homology sequence that arises by tensoring it with $\mathcal{S}\cong\mathcal{R}/I$ over $\mathcal{R}$:

$$0\longrightarrow\mathrm{Tor}_1^{\mathcal{R}}(\mathcal{M},\mathcal{S})\cong(I\mathcal{R}^n\cap\mathcal{N})/I\mathcal{N}\longrightarrow\mathcal{N}/I\mathcal{N}\longrightarrow(\mathcal{R}/I)^n\longrightarrow\mathcal{M}/I\mathcal{M}\longrightarrow 0.$$

By the Artin-Rees lemma, there exists $m\in\mathbb{N}$ such that for $l\gg 0$

$$\tau^l(I\mathcal{R}^n\cap\mathcal{N})\subset I^{q^l}\mathcal{R}^n\cap\mathcal{N}=I^{q^l-m}(I^m\mathcal{R}^n\cap\mathcal{N})\subset I\mathcal{N}.$$

This shows that $\mathrm{Tor}_1^{\mathcal{R}}(\mathcal{M},\mathcal{S})$ is nilpotent and therefore finishes the proof of the theorem. ∎

Definition 8.10 (Tensor product) *For any rigid τ-sheaves $\underline{\tilde{\mathcal{F}}}$ and $\underline{\tilde{\mathcal{G}}}$ on $\mathfrak{X}$ over $\mathcal{B}$, we let $\underline{\tilde{\mathcal{F}}} \otimes \underline{\tilde{\mathcal{G}}}$ denote the rigid τ-sheaf consisting of $\tilde{\mathcal{F}} \otimes_{\mathcal{O}_{\mathfrak{X}\times\mathfrak{B}}} \tilde{\mathcal{G}}$ and the composite homomorphism*

$$\begin{tikzcd} \sigma^*_{\mathfrak{X}/\mathcal{B}}(\tilde{\mathcal{F}} \otimes \tilde{\mathcal{G}}) \arrow[r, "\tau_{\tilde{\mathcal{F}}\otimes\tilde{\mathcal{G}}}"] \arrow[d, equal] & \tilde{\mathcal{F}} \otimes \tilde{\mathcal{G}}. \\ (\sigma^*_{\mathfrak{X}/\mathcal{B}}\tilde{\mathcal{F}}) \otimes (\sigma^*_{\mathfrak{X}/\mathcal{B}}\tilde{\mathcal{G}}) \arrow[ur, "(\tau_{\tilde{\mathcal{F}}})\otimes\tau_{\tilde{\mathcal{G}}}"'] & \end{tikzcd}$$

With the usual tensor product of homomorphisms this defines a $\mathcal{B}^\sigma$-bilinear bi-functor

$$\otimes \colon \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathcal{B}) \times \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathcal{B})$$

Its exactness properties are governed by the associated Tor-objects, cf. [4], § 3.

Proposition 8.11 *(a) If $\underline{\tilde{\mathcal{F}}}$ or $\underline{\tilde{\mathcal{G}}}$ is nilpotent, then so is $\underline{\tilde{\mathcal{F}}} \otimes \underline{\tilde{\mathcal{G}}}$ and $\mathrm{Tor}_i(\underline{\tilde{\mathcal{F}}}, \underline{\tilde{\mathcal{G}}})$ for every i.*

(b) If φ and ψ are nil-isomorphisms, then so is $\varphi \otimes \psi$.

(c) The above functor on τ-sheaves induces a unique $\mathcal{B}^\sigma$-bilinear bi-functor

$$\otimes \colon \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B}) \times \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B})$$

(d) The functor $\otimes$ is right exact in each variable.

Definition 8.12 *We let $\underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathcal{B}}$ denote the rigid τ-sheaf on $\mathfrak{X}$ over $\mathcal{B}$ consisting of the structure sheaf $\mathcal{O}_{\mathfrak{X}\times\mathfrak{B}}$ and the natural isomorphism*

$$\sigma^*_{\mathfrak{X}/\mathcal{B}}\mathcal{O}_{\mathfrak{X}\times\mathfrak{B}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}\times\mathfrak{B}}.$$

By $\underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathcal{B}}$ we also denote the corresponding unit crystal *on $\mathfrak{X}$ over $\mathcal{B}$.*

The canonical isomorphism $\underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathcal{B}} \otimes \underline{\tilde{\mathcal{G}}} \cong \underline{\tilde{\mathcal{G}}}$ allows us to view this as a unit object in $\widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B})$.

The following compatibilities, both on rigid τ-sheaves and on rigid crystals, are obvious by construction:

Proposition 8.13 *(a) $\underline{\tilde{\mathcal{F}}} \otimes (\underline{\tilde{\mathcal{G}}} \otimes \underline{\tilde{\mathcal{H}}}) \cong (\underline{\tilde{\mathcal{F}}} \otimes \underline{\tilde{\mathcal{G}}}) \otimes \underline{\tilde{\mathcal{H}}}$.*

(b) $f^*(\underline{\tilde{\mathcal{F}}} \otimes \underline{\tilde{\mathcal{G}}}) \cong f^*\underline{\tilde{\mathcal{F}}} \otimes f^*\underline{\tilde{\mathcal{G}}}$.

Definition 8.14 *A rigid crystal $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{X}$ is called* locally free*, if there exists an admissible affinoid cover $\mathfrak{U}_i = \operatorname{Spec} \mathcal{R}_i$ of $\mathfrak{X}$ and $\mathfrak{V}_j = \operatorname{Spec} \mathcal{S}_j$ of $\mathfrak{B}$ such that for each i, j the restriction $\underline{\tilde{\mathcal{F}}}_{|\mathfrak{U}_i \times \mathfrak{V}_j}$ is representable by a τ-sheaf whose underlying sheaf is associated to a projective module of finite rank over $\mathcal{R}_i \otimes_k \mathcal{S}_j$.*

Clearly locally free rigid crystals are acyclic for $\otimes$:

Proposition 8.15 *Let $\underline{\tilde{\mathcal{F}}}$ be a locally free rigid crystal on $\mathfrak{X}$. Then the functor*

$$\_\_ \otimes \underline{\tilde{\mathcal{F}}} \colon \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B})$$

is exact.

Definition 8.16 *A rigid τ-sheaf $\underline{\tilde{\mathcal{F}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathfrak{B})$ is called* of pullback type *if there exists a coherent rigid sheaf $\tilde{\mathcal{F}}_0$ on $\mathfrak{X}$ for which $\tilde{\mathcal{F}} = \mathrm{pr}_1^* \tilde{\mathcal{F}}_0$.*

A rigid crystal is called of pullback type, *if it can be represented by a rigid τ-sheaf of pullback type.*

Definition 8.17 *For an ideal sheaf $\mathcal{I}$ on $\mathfrak{X}$ and a rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ we define $\mathcal{I}\tilde{\mathcal{F}}$ as the image of $\mathcal{I} \otimes_{\mathcal{O}_\mathfrak{X}} \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$. The map $\tau_{\tilde{\mathcal{F}}}$ clearly induces a map $\sigma^*_{\mathfrak{X}/\mathcal{B}}(\mathcal{I}\tilde{\mathcal{F}}) \to (\mathcal{I}\tilde{\mathcal{F}})$, and the resulting rigid τ-sheaf is denoted $\mathcal{I}\underline{\tilde{\mathcal{F}}}$.*

As in the algebraic case, one expects in fact that any rigid crystal of pullback type is acyclic for $\otimes$, cf. Def. 8.16. We only provide the following result, which is a direct consequence of Theorem 8.9 and its proof.

Proposition 8.18 *Let $\mathcal{I}$ be an ideal sheaf on $\mathfrak{X}$. Then for any rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ we have:*

(a) The map $(\mathcal{I}\tilde{\underline{\mathbb{1}}}_{\mathcal{X}/\mathcal{B}}) \otimes \underline{\tilde{\mathcal{F}}} \to \mathcal{I}\underline{\tilde{\mathcal{F}}}$ is an nil-isomorphism.

(b) The rigid τ-sheaf $\mathrm{Tor}^i(\_\_, \mathcal{I}\tilde{\underline{\mathbb{1}}}_{\mathcal{X}/CB}) = 0$ is nilpotent whenever $i \neq 0$.

Let $h \colon \mathfrak{B}' \to \mathfrak{B}$ be a map of rigid spaces over L.

Definition 8.19 (Change of coefficients) *For any rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{X}$ over $\mathcal{B}$ we let $\underline{\tilde{\mathcal{F}}} \otimes_\mathcal{B} \mathcal{B}'$ denote the τ-sheaf on $\mathfrak{X}$ over $\mathcal{B}'$ consisting of $(\mathrm{id} \times h)^* \tilde{\mathcal{F}}$ together with the composite homomorphism*

$$\begin{array}{ccc} \sigma^*_{\mathfrak{X}/\mathcal{B}}(\tilde{\mathcal{F}} \otimes_\mathcal{B} \mathcal{B}') & \xrightarrow{\tau_{\tilde{\mathcal{F}} \otimes_\mathcal{B} \mathcal{B}'}} & \tilde{\mathcal{F}} \otimes_\mathcal{B} \mathcal{B}'. \\ \| & \nearrow{\scriptstyle (\tau_{\tilde{\mathcal{F}}}) \otimes_\mathcal{B} \mathcal{B}'} & \\ (\sigma^*_{\mathfrak{X}/\mathcal{B}} \tilde{\mathcal{F}}) \otimes_\mathcal{B} \mathcal{B}' & & \end{array}$$

For any morphism $\varphi \colon \underline{\tilde{\mathcal{F}}} \to \underline{\tilde{\mathcal{F}}}'$ we write $\varphi \otimes_\mathcal{B} \mathcal{B}'$ for the induced morphism.

This defines a $\mathcal{B}^\sigma$-linear functor

$$\_\_ \otimes_\mathcal{B} \mathcal{B}' \colon \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathcal{B}').$$

Our notation is meant to emphasize the role of $\mathcal{B}$ and $\mathcal{B}'$ as 'rings' of coefficients. Again, $\_\_ \otimes_\mathcal{B} \mathcal{B}'$ is not necessarily an exact functor and one can define left derived functors $\mathrm{Tor}_i^\mathcal{B}(\_\_, \mathcal{B}')$ to measure the deviation from exactness.

Proposition 8.20 *(a) If $\underline{\tilde{\mathcal{F}}}$ is locally nilpotent, then so is $\underline{\tilde{\mathcal{F}}} \otimes_\mathcal{B} \mathcal{B}'$.*

(b) If φ is a nil-isomorphism, then so is $\varphi \otimes 1$.

(c) The above functor on rigid τ-sheaves induces a unique $\mathcal{B}^\sigma$-linear functor

$$\_\_ \otimes_\mathcal{B} \mathcal{B}' : \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B}').$$

(d) The functors $\otimes_\mathcal{B} \mathcal{B}'$ on rigid τ-sheaves and crystals are right exact.

Proposition 8.21 *Let $\underline{\tilde{\mathcal{F}}}$ be a rigid crystal on $\mathfrak{X}$ of pullback type. Then for any h and any $i > 0$ one has $\mathrm{Tor}_i^\mathcal{B}(\underline{\tilde{\mathcal{F}}}, \mathcal{B}') = 0$.*

The following compatibilities, both on τ-sheaves and on crystals, are obvious by construction:

Proposition 8.22 *(a)* $f^*(\underline{\tilde{\mathcal{F}}} \otimes_{\mathcal{B}} \mathcal{B}') \cong f^*\underline{\tilde{\mathcal{F}}} \otimes_{\mathcal{B}} \mathcal{B}'$.

(b) $(\underline{\tilde{\mathcal{F}}} \otimes_{\mathcal{B}} \mathcal{B}') \otimes_{\mathcal{B}'} \mathcal{B}'' \cong \underline{\tilde{\mathcal{F}}} \otimes_{\mathcal{B}} \mathcal{B}''$.

Definition 8.23 (direct image) *Let f be a proper morphism. For any rigid τ-sheaf $\underline{\tilde{\mathcal{G}}}$ on $\mathfrak{Y}$ we let $R^i f_* \underline{\tilde{\mathcal{G}}}$ denote the rigid τ-sheaf on $\mathfrak{X}$ consisting of $R^i(f \times \mathrm{id})_* \tilde{\mathcal{G}}$ and the composite homomorphism*

$$\begin{tikzcd}
\sigma^*_{\mathfrak{X}/\mathcal{B}} R^i(f \times \mathrm{id})_* \tilde{\mathcal{G}} \arrow[rr, "\tau_{R^i f_* \tilde{\mathcal{G}}}"] \arrow[d, "\text{base change}"'] & & R^i(f \times \mathrm{id})_* \tilde{\mathcal{G}}. \\
R^i(f \times \mathrm{id})_* \sigma_{\mathfrak{X}}/\mathcal{B}^* \tilde{\mathcal{G}} \arrow[urr, "R^i(f\times\mathrm{id})_*\tau_{\tilde{\mathcal{G}}}"'] & &
\end{tikzcd}$$

For any homomorphism $\psi \colon \underline{\tilde{\mathcal{G}}} \to \underline{\tilde{\mathcal{G}}}'$ we abbreviate $R^i f_* \psi := R^i(f \times \mathrm{id})_*(\psi)$.

This defines $\mathcal{B}^\sigma$-linear functors

$$R^i f_* \colon \widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathcal{B}).$$

If $i = 0$, we write f_* for $R^0 f_*$. The functor f_* is clearly left exact, and when f is finite, it is exact and $R^i f_* = 0$ for $i > 0$.

Proposition 8.24 *Suppose f is proper.*

(a) If $\underline{\tilde{\mathcal{G}}}$ is nilpotent, then so are the $R^i f_* \underline{\tilde{\mathcal{G}}}$.

(b) If ψ is a nil-isomorphism, then so are the $R^i f_* \psi$.

(c) The above functors on rigid τ-sheaves induce unique $\mathcal{B}^\sigma$-linear functors

$$R^i f_* \colon \widetilde{\mathbf{Crys}}(\mathfrak{Y}, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B}).$$

(d) The functor f_* is left exact. When f is finite, it is exact and $R^i f_* = 0$ for $i > 0$.

The proof uses the fact that for a proper morphism f and an affinoid $\mathfrak{U}$ of $\mathfrak{X}$, the rigid space $f^{-1}\mathfrak{U}$ has a finite cover by affinoids. Based on the observation in Remark 8.3 it is then easy to adapt the proof in the algebraic context, given in [4], § 3.

We also have:

Proposition 8.25 *For any morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ there is a spectral sequence*

$$E_2^{ij} = R^i f_* R^j g_* \Longrightarrow R^{i+j}(fg)_*$$

induced from that of coherent cohomology.

Proposition 8.26 *If f is proper, the functor f_* is right adjoint to the functor f^*.*

8.3 Extension by zero

A rigid space $\mathfrak{X}$ is called *algebraic*, if we have $\mathfrak{X} \cong X^{\mathrm{rig}}$ for some scheme which is of finite type over L. A morphism of rigid spaces $\mathfrak{Y} \to \mathfrak{X}$ is called *algebraic*, if there exists a morphism of schemes $Y \to X$ which are of finite type over L, such that its rigidification is $\mathfrak{Y} \to \mathfrak{X}$. The morphism $\mathfrak{Y} \to \mathfrak{X}$ is called an *algebraic compactification*, if $Y \to X$ is an open immersion and X is proper. Note that any algebraic rigid space has an algebraic compactification.

From now on, for the remainder of this subsection, we fix an algebraic rigid space $\mathfrak{U}$ and an algebraic open immersion $j \colon \mathfrak{U} \to \mathfrak{X}$ of rigid spaces. We choose a closed algebraic complement $i \colon \mathfrak{Z} \to \mathfrak{X}$ of $\mathfrak{U}$ with ideal sheaf $\mathcal{I}$. Although $\mathcal{I}$ is far from unique, this will cause no difficulties.

In the algebraic context, it is possible to define for any open immersion a functor 'extension by zero' on the category of crystals. This relies on the following two facts: a) Every coherent sheaf on an open subscheme has a coherent extension. b) Along a closed complement of the open immersion, the operation τ can only have a pole of finite order. Both of these facts fail in the analytic context. One can have coherent sheaves over the punctured disc which have no coherent extension and one may define a τ-operation on the punctured disc which has an essential singularity at the origin. Therefore we can only expect to have an extension by zero for those rigid τ-sheaves which do admit an extension. The following two subsections describe a possible approach to a functor 'extension by zero' in the analytic context, as well as 'direct image with compact supports', for algebraic rigid spaces.

Lemma 8.27 *Suppose we have a commutative diagram*

$$\begin{tikzcd} & \mathfrak{Y}'' \arrow[d, "g"] \\ \mathfrak{U} \arrow[ur, "j''"] \arrow[r, "j'"'] & \mathfrak{Y}', \end{tikzcd}$$

where j', j'' are algebraic compactifications. Then we have

$$j'^*(\widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}', \mathcal{B})) = j''^*(\widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}'', \mathcal{B})) \subset \widetilde{\mathbf{Coh}}_\tau(\mathfrak{U}, \mathcal{B}).$$

In particular $j'^*(\widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}', \mathcal{B})) \subset \widetilde{\mathbf{Coh}}_\tau(\mathfrak{U}, \mathcal{B})$ is independent of the chosen algebraic compactification $j' \colon \mathfrak{U} \to \mathfrak{Y}$.

Proof: Since $j'^* \cong j''^* g^*$, the left hand side is clearly a subset of the right hand side. For the opposite inclusion, observe that $j''^* \cong j'^* g_*$ and that g_* preserves coherence, because g is proper. This proves the first assertion.

The second assertion immediately follows from the first, because any two algebraic compactifications are dominated by a third. ∎

By the above lemma, the following definition makes sense.

Definition 8.28 *Let $j' \colon \mathfrak{U} \to \mathfrak{Y}$ be any algebraic compactification. We define*

$$\widetilde{\mathbf{Coh}}{}^{e}_{\tau}(\mathfrak{U}, \mathcal{B}) := j'^*(\widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}, \mathcal{B})) \subset \widetilde{\mathbf{Coh}}_\tau(\mathfrak{U}, \mathcal{B})$$

as the category of extendible rigid τ-sheaves on $\mathfrak{U}$ over $\mathcal{B}$,.

There are examples that show that $\widetilde{\mathbf{Coh}}{}^{e}_{\tau}(\mathfrak{U}, \mathcal{B})$ is *not* a full subcategory of $\widetilde{\mathbf{Coh}}_\tau(\mathfrak{U}, \mathcal{B})$. In Proposition 8.49, we will show that any rigid τ-sheaf which arises via rigidification from an algebraic one is extendible.

The following is a direct consequence of the above definition.

Proposition 8.29 *The category* $\widetilde{\mathbf{Coh}}^{\,e}_{\tau}(\mathfrak{U},\mathcal{B})$ *is a* $\mathcal{B}^\sigma$*-linear abelian subcategory of* $\widetilde{\mathbf{Coh}}_{\tau}(\mathfrak{U},\mathcal{B})$*.*

Definition 8.30 *By* $\widetilde{\mathbf{Crys}}^{e}(\mathfrak{U},\mathcal{B})$ *we denote the essential image under the localization map* $\widetilde{\mathbf{Coh}}_{\tau}(\mathfrak{U},\mathcal{B}) \to \widetilde{\mathbf{Crys}}(\mathfrak{U},\mathcal{B})$*. Objects of* $\widetilde{\mathbf{Crys}}^{e}(\mathfrak{U},\mathcal{B})$ *are called* extendible crystals on $\mathfrak{X}$ over $\mathcal{B}$*.*

Clearly one may also define $\widetilde{\mathbf{Crys}}^{e}(\mathfrak{U},\mathcal{B})$ in the following way:

Proposition 8.31 *Let* $j' : \mathfrak{U} \to \mathfrak{Y}$ *be any algebraic compactification. Then* $\widetilde{\mathbf{Crys}}^{e}(\mathfrak{U},\mathcal{B}) = (j')^* \widetilde{\mathbf{Crys}}(\mathfrak{Y},\mathcal{B})$*.*

It is also possible to localize $\widetilde{\mathbf{Coh}}^{\,e}_{\tau}(\mathfrak{U},\mathcal{B})$ at its subset $\mathcal{S}_e$ of nil-isomorphisms, which is localizing. We do not know whether $\mathcal{S}_e^{-1}\widetilde{\mathbf{Coh}}^{\,e}_{\tau}(\mathfrak{U},\mathcal{B}) \to \widetilde{\mathbf{Crys}}^{e}(\mathfrak{U},\mathcal{B})$ is faithful.

Now that we have clarified some of the basic properties of extendible τ-sheaves, we will explain how to use them to define a functor 'extension by zero'.

Proposition 8.32 *For any extendible rigid* τ*-sheaf* $\underline{\tilde{\mathcal{F}}}$ *on* $\mathfrak{U}$ *there exists an extendible rigid* τ*-sheaf* $\underline{\tilde{\mathcal{G}}}$ *on* $\mathfrak{X}$ *such that* $\underline{\tilde{\mathcal{F}}} \cong j^*\underline{\tilde{\mathcal{G}}}$ *and* $i^*\underline{\tilde{\mathcal{F}}}$ *is nilpotent.*

PROOF: Let $j'' : \mathfrak{X} \to \mathfrak{Y}$ be any algebraic compactification, which exists because $\mathfrak{X}$ is algebraic, and set $j' := j''j$. Let $\mathcal{I}''$ be the ideal sheaf of an algebraic complement of j' and $\underline{\tilde{\mathcal{G}}}'$ any extension of $\underline{\tilde{\mathcal{F}}}$ to $\mathfrak{Y}$. One easily verifies that the extendible sheaf $j''^*(\mathcal{I}''\underline{\tilde{\mathcal{G}}}')$ satisfies all the required properties. ∎

Lemma 8.33 *Suppose* $\mathfrak{X}$ *has a finite cover by affiniods. Let* $\delta : \underline{\tilde{\mathcal{G}}} \to \underline{\tilde{\mathcal{G}}}'$ *be a morphism in* $\widetilde{\mathbf{Coh}}_{\tau}(\mathfrak{X},\mathfrak{B})$ *with* $j^*\delta = 0$ *and assume that* $i^*\underline{\tilde{\mathcal{G}}}'$ *is nilpotent. Then* $\operatorname{Im}(\delta)$ *is nilpotent.*

PROOF: Because $\mathfrak{X}$ has a finite cover by affinoids, there exists $m \in \mathbb{N}$ such that $\mathcal{I}^m \operatorname{Im}(\delta) = 0$. By changing i if necessary, we may assume that $m = 1$, so that

$$\operatorname{Im}(\delta) \cong i_*i^* \operatorname{Im}(\delta) \subset i_*i^*\underline{\tilde{\mathcal{G}}}'.$$

But $i^*\underline{\tilde{\mathcal{G}}}'$ is nilpotent, and hence the same holds for its sub-τ-sheaf $i^*\operatorname{Im}(\delta)$. ∎

Proposition 8.34 *Suppose* $\underline{\tilde{\mathcal{F}}}$ *and* $\underline{\tilde{\mathcal{H}}}$ *are extendible rigid* τ*-sheaves on* $\mathfrak{U}$ *and* $\mathfrak{X}$*, respectively. Then for any morphism* $\varphi : \underline{\tilde{\mathcal{F}}} \to j^*\underline{\tilde{\mathcal{H}}}$ *in* $\widetilde{\mathbf{Coh}}^{\,e}_{\tau}(\mathfrak{U},\mathcal{B})$ *and any extension* $\underline{\tilde{\mathcal{G}}}$ *as in Proposition 8.32, there exists* $n \in \mathbb{N}$ *such that* φ *extends to a homomorphism* $\tilde{\varphi} : \mathcal{I}^n\underline{\tilde{\mathcal{G}}} \to \underline{\tilde{\mathcal{H}}}$*.*

PROOF: Let $j' : \mathfrak{X} \to \mathfrak{Y}$ be an algebraic compactification, and choose a morphism $\varphi' : \underline{\tilde{\mathcal{F}}}' \to \underline{\tilde{\mathcal{H}}}'$ in $\widetilde{\mathbf{Coh}}_{\tau}(\mathfrak{Y},\mathcal{B})$ whose pullback along $j'' := j'j$ is φ. Let furthermore $\underline{\tilde{\mathcal{G}}}'$ be a τ-sheaf on $\mathfrak{Y}$ whose pullback along j' is $\underline{\tilde{\mathcal{G}}}$. Then $\underline{\tilde{\mathcal{G}}}'$ and $\underline{\tilde{\mathcal{H}}}'$ are coherent τ-subsheaves of $j''_*j^*\underline{\tilde{\mathcal{G}}}$. Using the coherence of $\underline{\tilde{\mathcal{G}}}' + \underline{\tilde{\mathcal{H}}}' \subset j''_*j^*\underline{\tilde{\mathcal{G}}}$, one easily shows that $\underline{\tilde{\mathcal{G}}}'' := \underline{\tilde{\mathcal{G}}}' \cap \underline{\tilde{\mathcal{H}}}'$ is a coherent extension of $j^*\underline{\tilde{\mathcal{G}}}$.

Let $\mathcal{I}''$ be the ideal sheaf of an algebraic complement $\mathfrak{Z}$ of j''. The cokernel of the inclusion $\underline{\tilde{\mathcal{G}}}'' \hookrightarrow \underline{\tilde{\mathcal{H}}}'$ is supported on $\mathfrak{Z}$, and so, by the previous lemma for $\delta = \mathrm{id}$, it is annihilated by a power of $\mathcal{I}''$, say $(\mathcal{I}'')^m$. Hence the restriction of φ' to $(\mathcal{I}'')^m\underline{\tilde{\mathcal{F}}}'$ takes its image in $\underline{\tilde{\mathcal{G}}}'' \subset \underline{\tilde{\mathcal{G}}}'$. Because $\mathcal{I}$ and $\mathcal{I}''$ arise from algebraic complements, there exists $n \in \mathbb{N}$ such that $\mathcal{I}^n \subset (j')^*(\mathcal{I}'')^m$. Therefore the restriction of φ' to $\mathfrak{X} \times \mathfrak{B}$ is the desired $\tilde{\varphi}$. ∎

Passing to the localized categories we now find:

Theorem 8.35 *(a) For every extendible rigid crystal $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{U}$ there exists $\underline{\tilde{\mathcal{G}}} \in \widetilde{\mathbf{Crys}}^e(\mathfrak{X}, \mathcal{B})$ such that $\underline{\tilde{\mathcal{F}}} \cong j^*\underline{\tilde{\mathcal{G}}}$ and $i^*\underline{\tilde{\mathcal{G}}}$ is zero in $\widetilde{\mathbf{Crys}}(\mathfrak{Z}, \mathcal{B})$.*

(b) The pair consisting of $\underline{\tilde{\mathcal{G}}}$ and the isomorphism $\underline{\tilde{\mathcal{F}}} \cong j^*\underline{\tilde{\mathcal{G}}}$ in (a) is unique up to unique isomorphism, and it depends functorially on $\underline{\tilde{\mathcal{F}}}$.

Proof: (a) is a reformulation of Proposition 8.32. To prove (b) we may assume that j is an algebraic compactification. Let us first show that the assignment

$$\underline{\tilde{\mathcal{F}}} \mapsto (\underline{\tilde{\mathcal{G}}}, j^*\underline{\tilde{\mathcal{G}}} \overset{\cong}{\to} \underline{\tilde{\mathcal{F}}})$$

depends functorially on $\underline{\tilde{\mathcal{F}}}$. For this, let $\varphi : \underline{\tilde{\mathcal{F}}} \to \underline{\tilde{\mathcal{F}}}'$ be a homomorphism of rigid τ-sheaves on $\mathfrak{U}$ and let $\underline{\tilde{\mathcal{G}}}$ and $\underline{\tilde{\mathcal{G}}}'$ be extensions as in 8.32. For any integer $n \in \mathbb{N}$, the inclusion $\mathcal{I}^n\underline{\tilde{\mathcal{G}}} \hookrightarrow \underline{\tilde{\mathcal{G}}}$ is a nil-isomorphism, because the cokernel is supported on $\mathfrak{Z}$ and annihilated by τ^n. Thus we may replace $\underline{\tilde{\mathcal{G}}}$ by $\mathcal{I}^n\underline{\tilde{\mathcal{G}}}$, if needed. By Proposition 8.34, we can thus achieve that φ extends to a homomorphism $\tilde{\varphi}\colon \underline{\tilde{\mathcal{G}}} \to \underline{\tilde{\mathcal{G}}}'$.

Lemma 8.33 shows that the resulting homomorphism in $\widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B})$ is unique, by considering the difference of any two extensions, i.e., that φ extends to a unique homomorphism $\underline{\tilde{\mathcal{G}}} \dashrightarrow \underline{\tilde{\mathcal{G}}}'$ in $\mathbf{Crys}(\mathfrak{X}, \mathcal{B})$. This is the desired functoriality. Applying it to the identity morphism on $\underline{\tilde{\mathcal{F}}}$ proves uniqueness, finishing the proof of (b). ∎

For every extendible rigid crystal $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{U}$ over $\mathcal{B}$, we choose $\underline{\tilde{\mathcal{G}}}$ as in Theorem 8.35 and denote it by $j_!\underline{\tilde{\mathcal{F}}}$. By the previous theorem this defines a $\mathcal{B}^\sigma$-linear functor, *extension by zero*,

$$j_!\colon \widetilde{\mathbf{Crys}}^e(\mathfrak{U}, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Crys}}^e(\mathfrak{X}, \mathcal{B}),$$

which is unique up to unique isomorphism.

Proposition 8.36 *The following assertions hold.*

(a) The functor $j_! : \widetilde{\mathbf{Crys}}^e(\mathfrak{U}, \mathcal{B}) \to \widetilde{\mathbf{Crys}}^e(\mathfrak{X}, \mathcal{B})$ is left adjoint to the functor $j^* : \widetilde{\mathbf{Crys}}^e(\mathfrak{X}, \mathcal{B}) \to \widetilde{\mathbf{Crys}}^e(\mathfrak{U}, \mathcal{B})$.

(b) The adjunction morphism $\mathrm{id} \to j^*j_!$ is an isomorphism on the category $\widetilde{\mathbf{Crys}}^e(\mathfrak{X}, \mathcal{B})$.

(c) The composite $i^*j_!$ is zero.

(d) The functor $j_!$ is exact.

(e) There is a natural exact sequence of functors on $\widetilde{\mathbf{Crys}}^e(\mathfrak{X}, \mathcal{B})$

$$0 \longrightarrow j_!j^* \longrightarrow \mathrm{id} \longrightarrow i_*i^* \longrightarrow 0.$$

Proof: The isomorphism in (b) exists by construction and is functorial by 8.35 (b), and its adjunction property (a) follows from Proposition 8.34. Assertion (c) is also clear by construction.

We now prove (d). It suffices to prove the assertion in the case where $\mathfrak{X}$ is proper. Let $0 \to \underline{\tilde{\mathcal{F}}}' \to \underline{\tilde{\mathcal{F}}} \to \underline{\tilde{\mathcal{F}}}'' \to 0$ be a short exact sequence in $\widetilde{\mathbf{Crys}}^e(\mathfrak{U}, \mathcal{B})$.

Using Proposition 8.34, we may represent the extension by zero of this short exact sequence by

$$\underline{\tilde{\mathcal{G}}}' \xrightarrow{\alpha} \underline{\tilde{\mathcal{G}}} \xrightarrow{\beta} \underline{\tilde{\mathcal{G}}}'' \tag{40}$$

in $\widetilde{\mathbf{Coh}}{}^{e}_{\tau}(\mathfrak{X}, \mathcal{B})$. By Lemma 8.33, the image of $\beta\alpha$ is nilpotent, and thus by replacing $\underline{\tilde{\mathcal{G}}}'$ by $\mathcal{I}^n\underline{\tilde{\mathcal{G}}}'$ for some $n > 0$, we may assume $\beta\alpha = 0$. The same lemma implies that $\mathrm{Ker}(\alpha)$ is nilpotent. Furthermore, as $\beta\alpha = 0$, the map α factors as

$$\underline{\tilde{\mathcal{G}}}' \longrightarrow \mathrm{Ker}(\beta) \hookrightarrow \underline{\tilde{\mathcal{G}}}.$$

So we have the four term exact sequence

$$0 \longrightarrow \mathrm{Ker}(\beta)/\operatorname{Im}(\alpha) \longrightarrow \underline{\tilde{\mathcal{G}}}/\operatorname{Im}(\alpha) \longrightarrow \underline{\tilde{\mathcal{G}}}'' \longrightarrow \operatorname{Coker}\beta \longrightarrow 0.$$

Again by Lemma 8.33, the rigid τ-sheaves $\mathrm{Ker}(\beta)/\operatorname{Im}(\alpha)$ and $\operatorname{Coker}\beta$ are nilpotent. Hence the sequence (40) is short exact in $\widetilde{\mathbf{Crys}}{}^{e}(\mathfrak{X}, \mathcal{B})$.

Part (e) is simple again, because for any $\underline{\tilde{\mathcal{G}}} \in \widetilde{\mathbf{Crys}}{}^{e}(\mathfrak{X}, \mathcal{B})$, the sequence in (e) is represented by the short exact sequence

$$0 \longrightarrow \mathcal{I}\underline{\tilde{\mathcal{G}}} \longrightarrow \underline{\tilde{\mathcal{G}}} \longrightarrow \underline{\tilde{\mathcal{G}}}/\mathcal{I}\underline{\tilde{\mathcal{G}}} \longrightarrow 0. \blacksquare$$

Finally, we list the following compatibilities in $\widetilde{\mathbf{Crys}}(\_\_, \mathcal{B})$:

Proposition 8.37 *Suppose* $\underline{\tilde{\mathcal{F}}}, \underline{\tilde{\mathcal{H}}} \in \widetilde{\mathbf{Crys}}{}^{e}(\mathfrak{U}, \mathcal{B})$ *and* $\underline{\tilde{\mathcal{G}}} \in \widetilde{\mathbf{Crys}}{}^{e}(\mathfrak{X}, \mathcal{B})$. *Then the following hold.*

(a) For open algebraic immersions $\mathfrak{U}' \overset{j'}{\hookrightarrow} \mathfrak{U} \overset{j}{\hookrightarrow} \mathfrak{X}$, *we have* $(jj')_! \cong j_! j'_!$.

(b) $j_!(\underline{\tilde{\mathcal{F}}} \otimes j^*\underline{\tilde{\mathcal{G}}}) \cong j_!\underline{\tilde{\mathcal{F}}} \otimes \underline{\tilde{\mathcal{G}}}$.

(c) $j_!(\underline{\tilde{\mathcal{F}}} \otimes \underline{\tilde{\mathcal{H}}}) \cong j_!\underline{\tilde{\mathcal{F}}} \otimes j_!\underline{\tilde{\mathcal{H}}}$.

(d) $j_!(\underline{\tilde{\mathcal{F}}} \otimes_{\mathcal{B}} \mathcal{B}') \cong j_!\underline{\tilde{\mathcal{F}}} \otimes_{\mathcal{B}} \mathcal{B}'$.

(e) For any algebraic morphism $g\colon \mathfrak{X}' \to \mathfrak{X}$ *consider the pullback diagram*

$$\begin{tikzcd} \mathfrak{U}' \arrow[r, hook, "j'"] \arrow[d, "g'"'] & \mathfrak{X}' \arrow[d, "g"] \\ \mathfrak{U} \arrow[r, hook, "j"] & \mathfrak{X}. \end{tikzcd}$$

Then the base change morphism $j'_! g'^* \underline{\tilde{\mathcal{F}}} \to g^* j_! \underline{\tilde{\mathcal{F}}}$ *is an isomorphism.*

Proof: In (a) through (d) one must show that the right hand side satisfies the characterization 8.35 (a) for the left hand side. Restricting both sides to $\mathfrak{U}$ yields clearly an isomorphism. To show the vanishing of the pullback to the closed complement one uses Lemma 8.33 and the various other foregoing compatibilities.

For (e), observe first, that using any compactification of $\mathfrak{X}$ and $\mathfrak{X}'$ one can construct a commutative diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "\tilde{j}'"] \arrow[d, "g"'] & \mathfrak{Y}' \arrow[d, "\tilde{g}"] \\ \mathfrak{X} \arrow[r, "\tilde{j}"] & \mathfrak{Y}, \end{tikzcd}$$

where $\mathfrak{Y}$ and $\mathfrak{Y}'$ are proper. Let $\tilde{\mathcal{I}}$ be the ideal sheaf of a complement of $\tilde{j}j$, and assume that $\mathcal{I} = \tilde{j}^*\tilde{\mathcal{I}}$, by changing i if necessary. Because the diagram in the statement of (e) is cartesian, $g^*\mathcal{I}$ is the ideal sheaf of a complement of $\mathfrak{U}'$. Let $\underline{\tilde{\mathcal{F}}}$ be an extendible sheaf on $\mathfrak{U}$ and $\underline{\tilde{\mathcal{G}}}'$ be an extension to $\mathfrak{Y}$. Then $\underline{\tilde{\mathcal{G}}} := \tilde{j}^*(\tilde{\mathcal{I}}\underline{\tilde{\mathcal{G}}}')$ represents $j_!\underline{\tilde{\mathcal{F}}}$, and hence $g^*\underline{\tilde{\mathcal{G}}}$ represents $g^*j_!\underline{\tilde{\mathcal{F}}}$. On the other hand, we have

$$g^*\tilde{j}^*(\tilde{\mathcal{I}}\underline{\tilde{\mathcal{G}}}') \cong g^*(\mathcal{I}\tilde{j}^*\underline{\tilde{\mathcal{G}}}') \cong (g^*\mathcal{I})(g^*\tilde{j}^*\underline{\tilde{\mathcal{G}}}') \cong (g^*\mathcal{I})(\tilde{j}')^*\tilde{g}^*\underline{\tilde{\mathcal{G}}}'.$$

The last expression clearly represents $j'_!g^*\underline{\mathcal{F}}$, and so (e) is proved. ∎

8.4 Direct image with compact support

In this subsection, we assume that $\mathfrak{X}$ is algebraic. No assumption on $\mathfrak{U}$ is made.

Definition 8.38 *A morphism $f\colon \mathfrak{U} \to \mathfrak{X}$ is called* algebraically compactifiable*, if there exists a diagram*

$$\begin{tikzcd} \mathfrak{U} \arrow[r, "j'"] \arrow[dr, "f"'] & \mathfrak{Y} \arrow[d, "\bar{f}"] \\ & \mathfrak{X}, \end{tikzcd} \tag{41}$$

where $\bar{f}$ is proper algebraic and j is an open algebraic immersion. The diagram (41) is called a compactification of *f.*

Since all objects in the above diagram are algebraic, $\mathfrak{Y}$ and $\mathfrak{X}$ have algebraic compactifications.

Definition 8.39 (direct image with compact supports) *Suppose $f\colon \mathfrak{U} \to \mathfrak{X}$ is algebraically compactifiable with a compactification as above. We define*

$$R^i f_! := R^i \bar{f}_* j_! \colon \widetilde{\mathbf{Crys}^e}(\mathfrak{U}, \mathcal{B}) \to \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathcal{B})$$

as the i-th right derived direct image with compact supports*.*

Strictly speaking $R^i f_!$ is not a right derived functor, it is only the composite of an exact with an i-th right derived functor.

As given, the above definition depends on the chosen algebraic compactification. The following result clarifies the situation.

Proposition 8.40 *The functors $R^i f_!$ are independent of the chosen algebraic compactification (in a functorial way) and take their image in $\widetilde{\mathbf{Crys}^e}(\mathfrak{X}, \mathcal{B})$.*

We first prove various lemmas:

Lemma 8.41 *Given a proper algebraic morphism $p\colon \mathfrak{Y}_2 \to \mathfrak{Y}_1$ between rigid spaces, there exists a pullback diagram*

$$\begin{tikzcd} \mathfrak{Y}_2 \arrow[r, "j'_2"] \arrow[d, "p"'] & \mathfrak{Z}_2 \arrow[d, "\bar{p}"] \\ \mathfrak{Y}_1 \arrow[r, "j'_1"] & \mathfrak{Z}_1, \end{tikzcd} \tag{42}$$

where the morphisms $j'_i\colon \mathfrak{Y}_i \to \mathfrak{Z}_i$ are algebraic compactifications and $\bar{p}$ is algebraic.

Proof: Because p is algebraic and all the assertions concern algebraic morphisms, it suffices to prove the assertion for schemes of finite type over L instead of rigid spaces. The scheme corresponding to a given rigid space is denoted by simply replacing the gothic by its corresponding latin letter. Morphisms will still have the same name.

So, let $Y_2 \xrightarrow{\tilde{j}} Z_2'$ and $Y_1 \xrightarrow{j_1'} Z_1$ be any compactifications. Define Z_2 as the Zariski closure of Y_2 in the product $Z_2' \times Z_1$ under $(\tilde{j}, j_1' p)$. The product has a canonical map to Z_1, receives a canonical map from Y_2, and is proper. Thus Z_2 is proper and there are an induced open immersion $j_2' \colon Y_2 \to Z_2$ and a morphism $\bar{p} \colon Z_2 \to Z_1$ such that the above diagram commutes.

To show that the so-constructed diagram is a pullback diagram, we need to show that the canonical morphism $g \colon Y_2 \to Y_1 \times_{Z_1} Z_2$ is an isomorphism. It is an open immersion, because j_2' as well as $Y_1 \times_{Z_1} Z_2 \to Z_2$ are open. At the same time it is proper, because p is separated and $Y_1 \times_{Z_1} Z_2 \to Y_1$ is proper, cf. [26], Cor. II.4.8. But any proper open immersion is an isomorphism. ∎

Lemma 8.42 *Suppose we are given a diagram as (42). Then*

$$j_1'^* R^i \bar{p}_* \cong R^i p_* j_2'^* \colon \widetilde{\mathbf{Coh}}_\tau(\mathfrak{Z}_2, \mathcal{B}) \longrightarrow \widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}_1, \mathcal{B}).$$

Proof: This follows from flat base change for rigid cohomology. The compatibility with τ is clear. ∎

Lemma 8.43 *Consider a pullback diagram*

$$\begin{tikzcd} \mathfrak{U}_2 \arrow[r, "j_2"] \arrow[d, "p'"'] & \mathfrak{Y}_2 \arrow[d, "p"'] \\ \mathfrak{U}_1 \arrow[r, "j_1"] & \mathfrak{Y}_1, \end{tikzcd}$$

where p, p' are proper algebraic, and j_1, j_2 are open algebraic immersions with dense image. Then

$$j_{1!} R^i p'_* \cong R^i p_* j_{2!} \colon \widetilde{\mathbf{Crys}}^e(\mathfrak{U}_1, \mathcal{B}) \to \widetilde{\mathbf{Crys}}^e(\mathfrak{Y}_2, \mathcal{B}) \quad \textit{for } i \geq 0.$$

Proof: By Lemma 8.41, we have a commutative diagram

$$\begin{tikzcd} \mathfrak{U}_2 \arrow[r, "j_2"] \arrow[d, "p'"'] & \mathfrak{Y}_2 \arrow[r, "j_2'"] \arrow[d, "p"] & \mathfrak{Z}_2 \arrow[d, "\bar{p}"] \\ \mathfrak{U}_1 \arrow[r, "j_1"] & \mathfrak{Y}_1 \arrow[r, "j_1'"] & \mathfrak{Z}_1, \end{tikzcd}$$

where both squares are pullback diagrams.

We claim that it suffices to prove the lemma for the large rectangular pullback square. Suppose we have shown $(j_1' j_1)_! R^i p'_* \cong R^i \bar{p} (j_2' j_2)_!$ for $i \geq 0$. Let us apply $j_1'^*$. Then

$$j_{1!} R^i p'_* \cong j_1'^* (j_1' j_1)_! R^i p'_* \cong j_1^* R^i \bar{p} (j_2' j_2)_! \overset{\text{Lem. 8.42}}{\cong} R^i p_* j_2'^* (j_2' j_2)_! \cong R^i p_* j_{2!},$$

and the claim is shown.

Thus from now on, we assume that $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ are proper. Let us fix a rigid extendible τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{U}$. We will make use of the following result from 'Residues and Duality' by Hartshorne, [27], which carries over to rigid coherent sheaves and is compatible with the $\sigma_{\mathfrak{X}/\mathcal{B}}$-linear morphisms, we consider: One has in the derived category of coherent rigid sheaves

$$Rp_*(\tilde{\mathcal{G}} \overset{L}{\otimes} Lp^*\tilde{\mathcal{H}}) \cong Rp_*(\tilde{\mathcal{G}}) \overset{L}{\otimes} \tilde{\mathcal{H}}.$$

This yields the following spectral sequences for rigid τ-sheaves $\underline{\tilde{\mathcal{G}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}_2, \mathcal{B})$ and $\underline{\tilde{\mathcal{H}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}_1, \mathcal{B})$:

$$\begin{aligned} E_2^{i,j} &= R^ip_*(H^j(\underline{\tilde{\mathcal{G}}} \overset{L}{\otimes} Lp^*\underline{\tilde{\mathcal{H}}})) &\implies\; & H^{i+j}(Rp_*(\underline{\tilde{\mathcal{G}}}) \overset{L}{\otimes} \underline{\tilde{\mathcal{H}}}), \\ E_2'^{\,i,j} &= \mathrm{Tor}_{-j}(R^ip_*\underline{\tilde{\mathcal{G}}}, \underline{\tilde{\mathcal{H}}}) &\implies\; & H^{i+j}(Rp_*(\underline{\tilde{\mathcal{G}}}) \overset{L}{\otimes} \underline{\tilde{\mathcal{H}}}), \\ E_2''^{\,i,j} &= \mathrm{Tor}_{-j}(\underline{\tilde{\mathcal{G}}}, L_{-i}p^*\underline{\tilde{\mathcal{H}}}) &\implies\; & H^{i+j}(\underline{\tilde{\mathcal{G}}} \overset{L}{\otimes} Lp^*\underline{\tilde{\mathcal{H}}}). \end{aligned}$$

Passing to the category of crystals, we can eliminate nilpotent terms. Furthermore we specialize $\underline{\tilde{\mathcal{H}}}$ to $\mathcal{I}\underline{\mathbb{1}}_{\mathfrak{Y}_1,\mathcal{B}}$, where $\mathcal{I}$ is the ideal sheaf of a complement of $\mathfrak{U}_1$ in $\mathfrak{Y}_1$. Also we take for $\underline{\tilde{\mathcal{G}}}$ some rigid τ-sheaf on $\mathfrak{Y}_2$ which extends $\underline{\tilde{\mathcal{F}}}$. Because the above diagram is cartesian, $p^*\mathcal{I}$ is the ideal sheaf of a complement of $\mathfrak{U}_2$ in $\mathfrak{Y}_2$. Proposition 8.18 yields

$$\mathrm{Tor}_i(\_\_, \mathcal{I}\underline{\mathbb{1}}_{\mathfrak{Y}_1,\mathcal{B}}) = \mathrm{Tor}_i(\_\_, (p^*\mathcal{I})\underline{\mathbb{1}}_{\mathfrak{Y}_2,\mathcal{B}}) = 0$$

for all $i \neq 0$, and isomorphisms $(\mathcal{I}\underline{\mathbb{1}}_{\mathfrak{Y}_1,\mathcal{B}}) \otimes \underline{\tilde{\mathcal{F}}'} \cong \mathcal{I}\underline{\tilde{\mathcal{F}}'}$ for $\underline{\tilde{\mathcal{F}}'} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{Y}_1, \mathcal{B})$, as well as $(p^*(\mathcal{I}\underline{\mathbb{1}}_{\mathfrak{Y}_1,\mathcal{B}})) \otimes \underline{\tilde{\mathcal{G}}} \cong (p^*\mathcal{I})\underline{\tilde{\mathcal{G}}}$ as crystals. Thus in $\mathbf{Crys}(\mathfrak{Y}_1, \mathcal{B})$ one has

$$\begin{aligned} E_2^{i,0} &= R^ip_*((p^*\mathcal{I})\underline{\tilde{\mathcal{G}}}), \\ E_2'^{\,i,0} &= \mathcal{I}R^ip_*\underline{\tilde{\mathcal{G}}}, \end{aligned}$$

and all E_2-terms with $j \neq 0$ are zero. The spectral sequence thus yields isomorphisms

$$R^ip_*((p^*\mathcal{I})\underline{\tilde{\mathcal{G}}}) \cong \mathcal{I}R^ip_*\underline{\tilde{\mathcal{G}}}$$

for all $i \geq 0$. By Lemma 8.42, it follows that $R^ip_*\underline{\tilde{\mathcal{G}}}$ is an extension of $R^ip'_*\underline{\tilde{\mathcal{F}}}$ for $i > 0$. Hence $\mathcal{I}R^ip_*\underline{\tilde{\mathcal{G}}}$ represents $j_{1!}R^ip_*\underline{\tilde{\mathcal{F}}}$. Observe also that $(p^*\mathcal{I})\underline{\tilde{\mathcal{G}}}$ represents $j_{2!}\underline{\tilde{\mathcal{F}}}$. Combining the last two identifications with the above isomorphism completes the proof of the lemma. ∎

Proof of Proposition 8.40: To see that the $R^if_!$ take their image in $\widetilde{\mathbf{Crys}^e}(\mathfrak{X})$, we apply Lemma 8.41 to have a commutative diagram of algebraic maps

$$\begin{tikzcd} \mathfrak{U} \arrow[r, "j"] \arrow[dr, "f"'] & \mathfrak{Y} \arrow[r, "j'"] \arrow[d, "\bar{f}"] & \mathfrak{Z} \arrow[d, "\bar{f}'"] \\ & \mathfrak{X} \arrow[r, "j''"] & \mathfrak{Z}', \end{tikzcd}$$

where the right hand square is cartesian, the horizontal maps are all open immersions, the vertical maps are proper, and $\mathfrak{Z}$, $\mathfrak{Z}'$ are proper. Let $\underline{\tilde{\mathcal{F}}} \in \widetilde{\mathbf{Crys}^e}(\mathfrak{U}, \mathcal{B})$ and $\underline{\tilde{\mathcal{G}}} \in \widetilde{\mathbf{Crys}}(\mathfrak{Z}, \mathcal{B})$ some extension. Let furthermore $\mathcal{I}'$ be the ideal sheaf of a complement of $j'j$. Then $j_!\underline{\tilde{\mathcal{F}}}$ is represented by $j'^*(\mathcal{I}'\underline{\tilde{\mathcal{G}}})$, and so we have

$$R^i\bar{f}_!\underline{\tilde{\mathcal{F}}} \cong R^i\bar{f}_*j'^*(\mathcal{I}'\underline{\tilde{\mathcal{G}}}) \cong j''^*R^i\bar{f}'_*(\mathcal{I}'\underline{\tilde{\mathcal{G}}}),$$

and it follows that $R^i \bar{f}_! \underline{\tilde{\mathcal{F}}}$ is extendible.

We now show the independence of $R^i f_!$ from the chosen algebraic compactification. Using standard techniques, e.g., [38], Ch. 6, § 3, any two algebraic compactifications are dominated by a third. Thus it suffices to consider the case where one algebraic compactification dominates another one. In this case, one needs to show that there is a functorial isomorphism between the two expressions for $R^i f_!$. Again one can follow the arguments in loc. cit. The proof ultimately rests on the spectral sequence for composition of direct image under proper morphisms, Prop. 8.25, and Lemma 8.43 applied to $\mathfrak{U}_1 = \mathfrak{U}_2 = \mathfrak{U}$ and $p' = \mathrm{id}$. ∎

As another application Lemma 8.43, we obtain:

Corollary 8.44 *Suppose $g\colon \mathfrak{Z} \to \mathfrak{Y}$ and $f : \mathfrak{Y} \to \mathfrak{X}$ are algebraically compactifiable. Then there is a spectral sequence*

$$R^i f_! R^j g_! \Longrightarrow R^{i+j}(fg)_! \colon \widetilde{\mathbf{Crys}}^e(\mathfrak{Z}, \mathcal{B}) \to \widetilde{\mathbf{Crys}}^e(\mathfrak{X}, \mathcal{B}).$$

Proof: From the given compactifications, one may construct a commutative diagram

$$\begin{tikzcd}
\mathfrak{Z} \arrow[r, hook, "j'"] \arrow[d, "g"'] & \mathfrak{Z}' \arrow[r, hook, "\bar{j}'"] \arrow[dl, "\bar{g}"'] \arrow[d, "g'"] & \mathfrak{Z}'' \arrow[dl, "\bar{g}'"] \\
\mathfrak{Y} \arrow[r, hook, "j"] \arrow[d, "f"'] & \mathfrak{Y}' \arrow[dl, "\bar{f}"] & \\
\mathfrak{X} & &
\end{tikzcd}$$

where the oblique maps are proper algebraic and the horizontal maps are open algebraic immersions, and such that the parallelogram formed by $\bar{j}'$, $\bar{g}'$, j, $\bar{g}$ is cartesian. By Lemma 8.43, we therefore have an isomorphism $j_! R^i \bar{g}_* \cong R^i \bar{g}'_* \bar{j}'_!$. But then we have

$$R^i f_! R^j g_! \overset{\text{def}}{=} R^i \bar{f} j_! R^j \bar{g} j'_! \cong R^i \bar{f}_* R^i \bar{g}'_* \bar{j}'_! j'_! \Longrightarrow R^{i+j}(\bar{f}\bar{g}')_*(j'\bar{j}')_! \overset{\text{def}}{=} R^{i+j}(fg)_!. \ ∎$$

8.5 Rigid τ-sheaves with B-coefficients

In this subsection, we develop some rudiments of a theory of rigid τ-sheaves with algebraic coefficients. Such objects arise naturally from Drinfeld-modules or Anderson's t-modules over a rigid base.

As we are not sure, whether the following natural definition can be found in the literature, we include it here for clarity.

Definition 8.45 *Let $\mathfrak{X}$ be a rigid space. We call a sheaf $\tilde{\mathcal{F}}$ of $\mathcal{O}_{\mathfrak{X}}$-modules* quasi-coherent*, if it is a direct limit of coherent $\mathcal{O}_{\mathfrak{X}}$-modules.*

Definition 8.46 *For B as above, we define the quasi-coherent sheaf of rings $B_{\mathfrak{X}}$ on $\mathfrak{X}$ as $\mathcal{O}_{\mathfrak{X}} \otimes B$, i.e., so that on an affinoid $\mathfrak{U}$ of $\mathfrak{X}$, we have $\Gamma(\mathfrak{U}, B_{\mathfrak{X}}) = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \otimes B$ with the obvious restriction (ringhomo-)morphisms.*

A coherent sheaf of $B_{\mathfrak{X}}$-modules *is a quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ such that over any affinoid $\mathfrak{U}$ of $\mathfrak{X}$, the sections $\Gamma(\mathfrak{U}, \mathcal{F})$ are a finitely generated module over $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \otimes B$ and such that the module structure is compatible with the restriction maps.*

Note that for any coherent sheaf $\mathcal{F}$ of $B_{\mathfrak{X}}$-modules the sections of the sheaf $\sigma_{\mathfrak{X}}^{*}\mathcal{F}$ on an affinoid $\mathfrak{U}$ are given by $\Gamma(\mathfrak{U},\mathcal{F})\otimes_{\Gamma(\mathfrak{U},\mathcal{O}_{\mathfrak{X}})}\Gamma(\mathfrak{U},\mathcal{O}_{\mathfrak{X}})$, where the tensor product is relative to the q-power map $\sigma_{\mathfrak{U}}\colon \Gamma(\mathfrak{U},\mathcal{O}_{\mathfrak{X}})\to\Gamma(\mathfrak{U},\mathcal{O}_{\mathfrak{X}}) : x\mapsto x^q$.

On an affinoid $\mathfrak{X}=\operatorname{Spm}\mathcal{R}$, the category of coherent $B_{\mathfrak{X}}$-modules is equivalent to the category of finitely generated $\mathcal{R}\otimes_k B$-modules M for which there exists a direct limit system M_i of finitely generated $\mathcal{R}$-modules such that $M=\varinjlim M_i$ as $\mathcal{R}$-modules.

As both of the properties in the previous paragraph are preserved under taking kernels, cokernels, images, coimages, the following is clear:

Proposition 8.47 *The category of coherent sheaves of $B_{\mathfrak{X}}$-modules is abelian.*

By $\sigma_{\mathfrak{X}/B}$, we denote the morphism of ringed spaces on $B_{\mathfrak{X}}$ which is the identity on the underlying topos, and $\sigma_{\mathfrak{X}}\otimes\mathrm{id}$ on $\mathcal{O}_{\mathfrak{X}}\otimes B$. Clearly $\sigma_{\mathfrak{X}/B}^{*}$ preserves coherent $B_{\mathfrak{X}}$-module. It will take the role played by $(\sigma\times\mathrm{id})^*$ in the algebraic setting.

Definition 8.48 *A* rigid τ-sheaf on $\mathfrak{X}$ over B, *is a pair $\underline{\tilde{\mathcal{F}}}:=(\tilde{\mathcal{F}},\tau_{\tilde{\mathcal{F}}})$ consisting of a coherent sheaf $\tilde{\mathcal{F}}$ of $B_{\mathfrak{X}}$-modules and a $B_{\mathfrak{X}}$-linear homomorphism*

$$(\sigma_{\mathfrak{X}/B})^{*}\tilde{\mathcal{F}}\xrightarrow{\tau_{\tilde{\mathcal{F}}}}\tilde{\mathcal{F}}.$$

The category of rigid τ-sheaves over B is denoted $\widetilde{\mathbf{Coh}}_{\tau}(\mathfrak{X},B)$. The morphisms are morphisms of sheaves of $B_{\mathfrak{X}}$-modules which are compatible with the morphism τ. The category is B-linear abelian, where kernel, cokernel, image and coimage are defined as for the underlying category of sheaves.

So if $\mathfrak{X}=\operatorname{Spm}\mathcal{R}$ is an affinoid, any rigid τ-sheaf on $\mathfrak{X}$ over B arises from a finitely generated τ-module over $\mathcal{R}\otimes_k B$. Occasionally, we will take this viewpoint.

Let $j:\mathfrak{U}\to\mathfrak{X}$ be an affinoid subdomain of $\mathfrak{X}$. Then by the restriction $\underline{\tilde{\mathcal{F}}}_{|\mathfrak{U}}$ or $j^*\underline{\tilde{\mathcal{F}}}$ of a rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{X}$ to $\mathfrak{U}$, we mean the pullback under j^* together with the induced morphism.

8.6 Some rigid sites

We will be interested in comparing crystals for the following sites. (Algebraic) crystals on X over A, rigid crystals on $\mathfrak{X}$ over A, and rigid crystals on $\mathfrak{X}$ over $\mathfrak{A}:=(\operatorname{Spec}L\otimes_k A)^{\mathrm{rig}}$ and over $\mathfrak{D}_A$, which is defined as follows: $\mathfrak{D}_A$ denotes the affinoid subset of points $x\in\mathfrak{A}$ such that for $n\to\infty$ the sequence $\tilde{\sigma}_{\mathfrak{A}}^{n}x$ does not converge to ∞. The rings corresponding to the above rigid spaces are $\mathcal{A}:=\Gamma(\mathfrak{A},\mathcal{O}_{\mathfrak{A}})$ and $\mathcal{D}_A:=\Gamma(\mathfrak{D}_A,\mathcal{O}_{\mathfrak{D}_A})$.

Note that one can also define $\mathfrak{D}_A$ as follows: Choose a finite flat morphism $k[T]\to A$. Let $L\langle\langle T\rangle\rangle$ denote the Tate-algebra over L. Then $\mathfrak{D}_A$ is isomorphic to $\operatorname{Spm}(L\langle\langle T\rangle\rangle\otimes_{k[T]}A)$. To see this, observe that a point of $\mathfrak{D}_A$ is given by a map $x:A\to L$ such that each $|x(a)|\leq 1$ for all $a\in A$, while a point of $\operatorname{Spm}(L\langle\langle T\rangle\rangle\otimes_{k[T]}A)$ consists of a map $x'\colon A\to L$ and an element $t\in L$, such that $|t|\leq 1$ and x' extends the map $k[T]\to L : T\mapsto t$. Because A is integral over $k[T]$ the two sets of data are equivalent. In particular for $A=k[t]$, one has $\mathcal{D}_A=L\langle\langle T\rangle\rangle$.

In analogy to Definition 8.46, we define for a noetherian scheme X over k the quasi-coherent sheaf of rings A_X by $U\mapsto\Gamma(U,\mathcal{O}_X)\otimes_k A$ for $U\subset X$ open. A coherent sheaf $\mathcal{F}$ of A_X-modules is defined as a quasi-coherent sheaf of $\mathcal{O}_X$-modules with a compatible action of A_X such that on any affine U of X the module $\Gamma(U, \mathcal{F})$ is finitely generated over $\Gamma(U, \mathcal{O}_X) \otimes A$. Analogous notions can be defined if X is replaced by a rigid space and A by $\mathfrak{A}$ or $\mathfrak{D}_A$, respectively.

Now the category of coherent sheaves of $\mathcal{O}_{X\times A}$-modules is equivalent to the category of coherent sheaves of A_X-modules. An analogous equivalence can be formulated for $\mathcal{O}_{\mathfrak{X}\times ?}$-modules, $? \in \{\mathfrak{A}, \mathfrak{D}_A\}$. Because any quasi-coherent sheaf of $\mathcal{O}_X$-modules is a direct limit of coherent sheaves of $\mathcal{O}_X$-modules, for any scheme X of finite type over L this yields a functor

$$\mathbf{Coh}_\tau(X, A) \longrightarrow \widetilde{\mathbf{Coh}}_\tau(X^{\mathrm{rig}}, A) : \underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{A\text{-rig}}.$$

Also for any rigid space $\mathfrak{X}$, the above defines a functor

$$\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, A) \longrightarrow \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathfrak{A}) : \underline{\tilde{\mathcal{F}}} \mapsto \underline{\tilde{\mathcal{F}}}^{\mathfrak{A}\text{-rig}}.$$

Furthermore the restriction of sheaves from $\mathfrak{X} \times \mathfrak{A}$ to $\mathfrak{X} \times \mathfrak{D}_A$ yields the functor

$$\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathfrak{A}) \longrightarrow \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathfrak{D}_A) : \underline{\tilde{\mathcal{F}}} \mapsto \underline{\tilde{\mathcal{F}}}^{\mathfrak{D}_A\text{-rig}}.$$

For $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, A)$ and $\underline{\tilde{\mathcal{F}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, A)$, we abbreviate the composite functors

$$\underline{\mathcal{F}}^{\mathfrak{A}\text{-rig}} := (\underline{\mathcal{F}}^{A\text{-rig}})^{\mathfrak{A}\text{-rig}}, \ \underline{\mathcal{F}}^{\mathfrak{D}_A\text{-rig}} := (\underline{\mathcal{F}}^{\mathfrak{A}\text{-rig}})^{\mathfrak{D}_A\text{-rig}}, \ \underline{\tilde{\mathcal{F}}}^{\mathfrak{D}_A\text{-rig}} := (\underline{\tilde{\mathcal{F}}}^{\mathfrak{A}\text{-rig}})^{\mathfrak{D}_A\text{-rig}}.$$

Note that on the underlying module-categories, each of the above functors can be defined on quasi-coherent sheaves and preserves quasi-coherence.

The characterization of nil-isomorphisms given in Propositions 7.6 and 8.6 shows that the above three functors induce functors

$$\mathbf{Crys}(X, A) \xrightarrow{\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{A\text{-rig}}} \widetilde{\mathbf{Crys}}(X^{\mathrm{rig}}, A)$$

$$\widetilde{\mathbf{Crys}}(\mathfrak{X}, A) \xrightarrow{\underline{\tilde{\mathcal{F}}} \mapsto \underline{\tilde{\mathcal{F}}}^{\mathfrak{A}\text{-rig}}} \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathfrak{A}) \xrightarrow{\underline{\tilde{\mathcal{G}}} \mapsto \underline{\tilde{\mathcal{G}}}^{\mathfrak{D}_A\text{-rig}}} \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathfrak{D}_A)$$

The first result on the above functors concerns the subclasses of extendible τ-sheaves.

Proposition 8.49 *The functor* $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{\mathfrak{A}\text{-rig}} : \mathbf{Crys}(X, A) \to \widetilde{\mathbf{Crys}}(X^{\mathrm{rig}}, \mathfrak{A})$ *takes its image in* $\widetilde{\mathbf{Crys}}^e(X^{\mathrm{rig}}, \mathfrak{A})$. *The functor*

$$\widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathfrak{A}) \to \widetilde{\mathbf{Crys}}(\mathfrak{X}, \mathfrak{D}_A) : \underline{\tilde{\mathcal{F}}} \mapsto \underline{\tilde{\mathcal{F}}}^{\mathfrak{D}_A\text{-rig}}$$

preserves the subcategory of extendible rigid sheaves.

Proof: For the first part, observe that by a result of Nagata, all schemes of finite type over L are compactifiable and by [4], § 3, every sheaf has an extension by zero to its compactification. It is simple to see, cf. also Theorem 8.50 below, that $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{A\text{-rig}}$ is compatible with j^* for an open algebraic immersion j. The result is now obvious.

The remaining assertion follows from the compatibility of the functor $\underline{\tilde{\mathcal{F}}} \mapsto \underline{\tilde{\mathcal{F}}}^{\mathfrak{D}_A\text{-rig}}$ with j^* for an open algebraic immersion j, see below. ∎

The following theorem summarizes the main compatibilities of the functors defined above.

Theorem 8.50 *The functors* $\mathbf{Coh}_\tau(X,A) \to \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},\mathfrak{A}) : \underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{\mathfrak{A}\text{-rig}}$ *and* $\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},\mathfrak{A}) \to \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},\mathfrak{D}_A) : \underline{\tilde{\mathcal{F}}} \mapsto \underline{\tilde{\mathcal{F}}}^{\mathfrak{D}_A\text{-rig}}$ *are compatible with*

(a) pullback,

(b) tensor product,

(c) change of coefficients,

(d) higher direct image under proper morphisms.

Considered as functors on crystals, the above two functors are compatible with (a)—(d) as well as the two functors

(e) extension by zero for extendible crystals,

(f) direct image with compact supports for extendible crystals and underlying spaces which are compactifiable.

The proof of the above theorem is essentially that given in [33] for a GAGA principle from schemes to rigid spaces, which in turn is based on Serre's original proof of a GAGA principle, [47]. The most difficult part in the proof is the compatibility with higher direct image functors under proper morphisms. To define a natural transformation for higher direct images, one uses Čech cohomology. For the convenience of the reader and because we will later use Čech cohomology as a computational device, we now recall some basic results on this, and then comment on the proof of the above theorem.

Let us first state the comparison theorem of Leray for Čech and derived functor cohomology in the algebraic and analytic context. For a module M over a ring R, we denote by M^{as} the associated quasi-coherent sheaf on $\operatorname{Spec} R$, and similarly for a module $\mathcal{M}$ on an affinoid ring $\mathcal{R}$, we denote by $\mathcal{M}^{\mathrm{as}}$ the associated quasi-coherent sheaf on $\operatorname{Spm}\mathcal{R}$.

Theorem 8.51 *Let* $f\colon Y \to X$ *be a morphism of schemes of finite type over* L *and* $\mathcal{F}$ *a quasi-coherent sheaf on* Y*. Assume* $X = \operatorname{Spec} R$ *and* $\{U_i\}$ *is a cover of* Y *such that* $H^j(U,\mathcal{F}) = 0$ *for all* $j > 0$ *and all finite intersections* U *of elements* U_i*. Then*

$$R^i f_*\mathcal{F} \cong \check{H}^i(\{U_i\},\mathcal{F})^{\mathrm{as}}.$$

Note that $H^j(U,\mathcal{F}) = 0$ whenever U is affine, so that one can use affine covers of Y to compute cohomology. Also note that the case where X is not affine can be treated by patching the above result for an affine cover of X.

Next we turn to the analogous result for rigid spaces.

Theorem 8.52 *Let* $f\colon \mathfrak{Y} \to \mathfrak{X}$ *be a morphism of schemes of rigid spaces over* L *and* $\tilde{\mathcal{F}}$ *a quasi-coherent sheaf on* $\mathfrak{Y}$*. Assume* $\mathfrak{X} = \operatorname{Spm}\mathcal{R}$ *and* $\{\mathfrak{U}_i\}$ *is an admissible cover of* $\mathfrak{Y}$ *such that* $H^j(\mathfrak{U},\tilde{\mathcal{F}}) = 0$ *for all* $j > 0$ *and all finite intersections* $\mathfrak{U}$ *of elements* $\mathfrak{U}_i$*. Then*

$$R^i f_*\tilde{\mathcal{F}} \cong \check{H}^i(\{\mathfrak{U}_i\},\tilde{\mathcal{F}})^{\mathrm{as}}.$$

By Tate's acyclicity theorem one has $H^i(\mathfrak{U},\tilde{\mathcal{F}}) = 0$ whenever $\mathfrak{U}$ is an affinoid. But more is true due to Kiehl's Theorem B, [32], Satz 2.4:

Theorem 8.53 *If* $\mathfrak{X}$ *is quasi-Stein, then for all quasi-coherent sheaves* $\tilde{\mathcal{F}}$ *and all* $i > 0$ *one has* $H^i(\mathfrak{X},\tilde{\mathcal{F}}) = 0$.

Observe that the product of quasi-Stein domains is again quasi-Stein. Furthermore for any affine scheme X, the space X^{rig} is quasi-Stein.

We now give the proof of Theorem 8.50.

PROOF: Note first that the result for $\underline{\tilde{\mathcal{F}}} \mapsto \underline{\tilde{\mathcal{F}}}^{\mathfrak{D}_A\text{-rig}}$ is obvious since restricting coefficients is compatible with any of the functors (a)—(f), and so from now on, we only consider the functor $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{\mathfrak{A}\text{-rig}}$. For τ-sheaves, except for (d) all the compatibilities are obvious.

To prove (d), let us first explain how to define the natural transformation that will lead to the compatibility of functors. For this, we consider coherent sheaves without a τ-action. Once the compatibility is shown for these, it is straightforward to extend it to τ-sheaves. We leave this extension procedure to the reader.

Let $\{U_i\}$ be an affine cover of Y and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{Y\times \operatorname{Spec} A}$-modules. By a patching argument, we may assume that X is affine. Denote the Čech complex with respect to $\{U_i \times \operatorname{Spec} A\}$ by $\mathcal{C}^\bullet(\{U_i\},\mathcal{F})$. Then we have

$$R^i f_* \mathcal{F} \cong H^i(\mathcal{C}^\bullet(\{U_i\},\mathcal{F}))^{\mathrm{as}}.$$

Similarly, we denote by $\mathcal{C}^\bullet(\{\mathfrak{U}_i\},\tilde{\mathcal{F}})$ the Čech complex of a rigid sheaf $\tilde{\mathcal{F}}$ of $\mathcal{O}_{\mathfrak{X}\times\mathfrak{A}}$-modules for an admissible cover $\mathfrak{U}_i \times \mathfrak{A}$ of $\mathfrak{X}\times\mathfrak{A}$. Because the rigid spaces U_i^{rig} and arbitrary intersection of these are Stein-domains, one has the formula

$$R^i f_*(\mathcal{F}^{\mathfrak{A}\text{-rig}}) \cong H^i(\mathcal{C}^\bullet(\{U_i^{\mathrm{rig}}\},\mathcal{F}^{\mathfrak{A}\text{-rig}}))^{\mathrm{as}}.$$

There is a functorial morphism of complexes

$$\Big(\mathcal{C}^\bullet(\{U_i\},\mathcal{F})\Big)^{\mathfrak{A}\text{-rig}} \longrightarrow \mathcal{C}^\bullet(\{U_i^{\mathrm{rig}}\},\mathcal{F}^{\mathfrak{A}\text{-rig}})$$

from the construction of $\mathcal{F} \mapsto \mathcal{F}^{\mathfrak{A}\text{-rig}}$. This yields the desired morphism

$$(R^i f_* \mathcal{F})^{\mathfrak{A}\text{-rig}} \longrightarrow R^i f_*(\mathcal{F}^{\mathfrak{A}\text{-rig}}).$$

To prove that this morphism is an isomorphism, one first considers the case of projective morphisms. There one can follow the proof given in [47], i.e., one first proves the result for the sheaves $\mathcal{O}_{\mathbb{P}^m_X} \otimes A$ by direct computation, then for the $\mathcal{O}_{\mathbb{P}^m_X}(n)_X \otimes A$ by some inductive argument, then for general coherent sheaves $\mathcal{F}$ on $\mathbb{P}^m_X \times \operatorname{Spec} A$ by using resolutions $0 \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}^m_X}(n) \otimes A \to \mathcal{F} \to 0$ and the fact that the cohomological dimension of the functors in question is n. The result easily extends to any projective morphism. To obtain the same result for general proper morphisms, one uses Chow's Lemma and a comparison of spectral sequences. For a complete argument in a similar situation see [33].

Passing from (rigid) τ-sheaves to crystals, immediately yields (a)–(d) for crystals. It is also clear that once (e) is proved, (f) is automatically true by (e) and (d). For this, let $\underline{\mathcal{F}}$ be a τ-sheaf on U, and $j : U \to X$ an open algebraic immersion. Choose a coherent extension $\underline{\mathcal{G}}$ to X and let $\mathcal{I}$ denote the ideal sheaf of a complement of j. Then $\mathcal{I}^{\mathrm{rig}}$ is the ideal sheaf of a complement of $j^{\mathrm{rig}} \colon U^{\mathrm{rig}} \to X^{\mathrm{rig}}$. Furthermore it is clear that

$$(\mathcal{I}\underline{\mathcal{G}})^{\mathfrak{A}\text{-rig}} \cong \mathcal{I}^{\mathrm{rig}}\underline{\mathcal{G}}^{\mathfrak{A}\text{-rig}}.$$

Because $\mathcal{I}\underline{\mathcal{G}}$ represents $j_!\underline{\mathcal{F}}$ and $\mathcal{I}^{\mathrm{rig}}\underline{\mathcal{G}}^{\mathfrak{A}\text{-rig}}$ represents $j_!\underline{\mathcal{F}}^{\mathrm{rig}}$, (e) is proved. ∎

Let us finally give some examples for rigid τ-sheaves, which will be important in the sequel. Following Example 7.7 (b), however, with a rigid analytic Drinfeld-module instead of an algebraic Drinfeld-module, one easily sees that to each rigid analytic Drinfeld module $(\underline{\varphi}, \mathcal{L})$ on $\mathfrak{X}$ of rank r, there is attached a locally free rigid τ-sheaf on $\mathfrak{X}$ over A. We denote it by $\underline{\tilde{\mathcal{M}}}(\underline{\varphi})$.

Definition 8.54 *For $\underline{\varphi}(\mathcal{K})$ as on page 44, we set $\underline{\tilde{\mathcal{F}}}(\mathcal{K}) := \underline{\tilde{\mathcal{M}}}(\underline{\varphi}(\mathcal{K}))$. Furthermore for the universal rigid analytic Drinfeld-module $\underline{\varphi}_{\mathcal{K}}^{\mathrm{rig}}$ for an admissible level structure $\mathcal{K}$, we define $\underline{\tilde{\mathcal{F}}}_{\mathcal{K}} := \underline{\tilde{\mathcal{M}}}(\underline{\varphi}_{\mathcal{K}})$.*

Proposition 8.55 *Suppose $\mathcal{K}$ is admissible. Then*

$$\mathrm{GL}_2(K)\backslash\underline{\tilde{\mathcal{F}}}(\mathcal{K}) \cong \underline{\tilde{\mathcal{F}}}_{\mathcal{K}}.$$

Furthermore one has

$$(\underline{\mathcal{F}}_{\mathcal{K}})^{A\text{-rig}} \cong \underline{\tilde{\mathcal{F}}}_{\mathcal{K}}.$$

PROOF: The first half of the proposition follows easily from the fact that $\underline{\varphi}_{\mathcal{K}}$ is constructed from $\underline{\varphi}(\mathcal{K})$ by quotienting the underlying space $\Omega_{\mathcal{K}}$ by $\mathrm{GL}_2(\bar{K})$, cf. Subsection 4.3. For the second one uses that the moduli space for analytic Drinfeld-modules of rank r with level $\mathcal{K}$-structure arises from the algebraic one by rigidification, and hence $\underline{\varphi}_{\mathcal{K}}^{\mathrm{rig}}$ arises from $\underline{\varphi}$ via the same procedure. This yields the result for the associated τ-sheaves. ∎

9 Auxiliary results on families of A-motives

9.1 Purity

Following Potemine, we define so-called pure Drinfeld-Anderson sheaves over any base X. The most basic example is Drinfeld's shtuka attached to any family of Drinfeld-modules over some base X. Our main result is that exterior, symmetric and tensor powers do preserve the category of pure Drinfeld-Anderson motives. This is a natural extension of results of Anderson and Potemine. It follows easily that the τ-sheaves $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} := \mathrm{Sym}^n \underline{\mathcal{F}}_{\mathcal{K}}$ arise from a pure Drinfeld-Anderson motive on $\mathfrak{M}_{\mathcal{K}}$ of rank $n+1$ and weight $n/2$.

The following is similar to [41] Def. 6.2.1:

Definition 9.1 *A* Drinfeld-Anderson datum $\mathcal{F}_{\mathrm{DA}} = (\mathcal{F}_i, \alpha_i, \beta_i, \mathrm{char})$ of rank r *consists of*

(a) locally free sheaves $\mathcal{F}_i$, $i \in \mathbb{Z}$ on $X \times C$ of rank r,

(b) monomorphisms $\beta_i \colon \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1}$, $i \in \mathbb{Z}$,

(c) monomorphisms $\alpha_i \colon (\sigma \times \mathrm{id})^* \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1}$, $i \in \mathbb{Z}$,

(d) and a morphism $\mathrm{char}\colon X \to \mathrm{Spec}\, A$

such that the following diagram commutes for all i

$$\begin{tikzcd} (\sigma \times \mathrm{id})^* \mathcal{F}_i \arrow[r, "\alpha_i"] \arrow[d, "(\sigma\times\mathrm{id})^*\beta_i"] & \mathcal{F}_{i+1} \arrow[d, "\beta_{i+1}"] \\ (\sigma \times \mathrm{id})^* \mathcal{F}_{i+1} \arrow[r, "\alpha_{i+1}"] & \mathcal{F}_{i+2} \end{tikzcd}$$

A pure Drinfeld-Anderson motive on X of rank r, dimension d and weight $w = d/r$ *is a Drinfeld-Anderson datum $\mathcal{F}_{\mathrm{DA}}$ of rank r which is subject to the following conditions:*

(i) The sheaf $\mathrm{Coker}(\beta_i)$ *is supported on* $X \times \{\infty\}$ *and* $\mathrm{pr}_{1*}\mathrm{Coker}(\beta_i)$ *is locally free of rank d over X.*

(ii) The sheaf $\mathrm{Coker}(\alpha_i)$ *is supported on the graph of* char *in* $X \times \mathrm{Spec}\, A$ *and* $\mathrm{pr}_{1*}\mathrm{Coker}(\alpha_i)$ *is locally free of rank d on X.*

(iii) There exist integers u, v with $u/v = w$ such that for all i the map

$$\beta_{i+vd_\infty-1} \ldots \beta_{i+1}\beta_i \colon \mathcal{F}_i \to \mathcal{F}_{i+vd_\infty}$$

identifies $\mathcal{F}_i$ with the subsheaf $\mathcal{F}_{i+vd_\infty}(-X \times \{u\infty\})$ of $\mathcal{F}_{i+vd_\infty}$.

Let π_∞ be any uniformizer of A_∞. The following definition is from [1], § 1.9 and [41], § 6.1, respectively.

Definition 9.2 *A* pure A-motive of rank r and weight w on a field F *is an A-motive M over $F \otimes A\{\tau\}$, which is projective over $F \otimes A$ of rank r, such that there exists an $F\hat{\otimes}_A A_\infty$-lattice W contained in $M\hat{\otimes}_A K_\infty$ which satisfies the following: For suitable $u, v \in \mathbb{N}$ with $w = u/v$ one has $\tau^{vd_\infty}(W) = \pi_\infty^{-u} W$, where τ is the action on $M\hat{\otimes}_A K_\infty$ induced from M.*

Remarks 9.3 (i) For any pure Drinfeld-Anderson motive $\mathcal{F}_{\mathrm{DA}}$ its restriction to $X \times \operatorname{Spec} A$ naturally defines a τ-sheaf on X, denoted by $\underline{\mathcal{F}}_{\mathrm{DA}} := (\mathcal{F}, \tau_{\mathcal{F}})$, as follows: Define $\mathcal{F} := \mathcal{F}_{0|X \times \operatorname{Spec} A}$ and

$$\tau_{\mathcal{F}} \colon (\sigma \times \mathrm{id})^* \mathcal{F} \overset{\alpha_{0|X\times\operatorname{Spec} A}}{\longrightarrow} \mathcal{F}_{1|X \times \operatorname{Spec} A} \overset{\beta^{-1}_{0|X\times\operatorname{Spec} A}}{\longrightarrow} \mathcal{F},$$

where we use that restrictions of the β_i to $X \times \operatorname{Spec} A$ are isomorphisms.

(ii) Let $X = \operatorname{Spec} F$ for an algebraically closed field F. By [41], § 6.1 and [1], § 1.9, there is a bijection between the set of pure Drinfeld-Anderson motives $\mathcal{F}_{\mathrm{DA}}$ on X and the set of pure A-motives (M, τ) on F such that $\tau^{-1}W \subset W$ for some (any) W. The equivalence is given by mapping $\mathcal{F}_{\mathrm{DA}}$ to (M, τ, W), where (M, τ) is defined as in (i) and W is the completion of the stalk of $\mathcal{F}_0$ at $\operatorname{Spec} F \times \{\infty\}$, so that W is a free $F\hat{\otimes}A_\infty$-module of rank r.

(iii) From (ii) it follows that all the geometric fibers of the τ-sheaf $\underline{\mathcal{F}}_{\mathrm{DA}}$, defined in (i), are pure A-motives of weight w and rank r, so that in particular $\underline{\mathcal{F}}_{\mathrm{DA}}$ defines a family of A-motives on X of characteristic char.

Proposition 9.4 *Let φ be a Drinfeld-module on X of rank r. Then to φ one has a naturally attached pure Drinfeld-Anderson motive $\mathcal{F}_{\mathrm{DA}}(\varphi)$ on $X \times C$ of rank r, dimension 1 and weight $1/r$, such that the τ-sheaf $\underline{\mathcal{F}}_{\mathrm{DA}}(\varphi)$ is canonically isomorphic to the τ-sheaf $\underline{M}(\varphi)$ in Example 7.7 (ii).*

Proof: The pure Drinfeld-Anderson motive $\mathcal{F}_{\mathrm{DA}}(\varphi)$ is precisely the shtuka attached to φ by Drinfeld. For some details on this construction we refer to [3], proof of Prop. 5.10, or [2], § 3.2. ∎

Suppose we are given pure Drinfeld-Anderson motives $\mathcal{F}_{\mathrm{DA}}$ and $\mathcal{F}'_{\mathrm{DA}}$ on X of ranks r, r' and weights w, w', respectively, which have the same characteristic. Then the tensor product defines in an obvious way a Drinfeld-Anderson datum denoted $\mathcal{F}_{\mathrm{DA}} \otimes \mathcal{F}'_{\mathrm{DA}}$ of rank rr'.

Lemma 9.5 *The datum $\mathcal{F}_{\mathrm{DA}} \otimes \mathcal{F}'_{\mathrm{DA}}$ defines a pure Drinfeld-Anderson motive of rank rr' and weight $w + w'$.*

Proof: We have to verify properties (i)–(iii) of the above definition: Because $\mathcal{F}_{\mathrm{DA}}$ and $\mathcal{F}'_{\mathrm{DA}}$ have the same characteristics, the cokernels of $\beta_i \otimes \beta'_i$ and $\alpha_i \otimes \alpha'_i$ are supported on $X \times \{\infty\}$ and the graph of char, respectively. So to check (i) and (ii) it remains to check that the pushforward under pr_1 of these cokernels is locally free on X and of dimension $(w + w')rr'$. Applying the Snake Lemma to the the diagram whose basic square is

$$\begin{tikzcd}
\mathcal{F}_i \otimes \mathcal{F}'_i \arrow[r, hook, "\beta_i\otimes\mathrm{id}"] \arrow[d, equal] & \mathcal{F}_{i+1} \otimes \mathcal{F}'_i \arrow[d, "\mathrm{id}\otimes\beta'_i"] \\
\mathcal{F}_i \otimes \mathcal{F}'_i \arrow[r, hook, "\beta_i\otimes\beta'_i"] & \mathcal{F}_{i+1} \otimes \mathcal{F}'_{i+1},
\end{tikzcd}$$

one obtains the short exact sequence

$$0 \to \operatorname{Coker} \beta_i \otimes \mathcal{F}'_i \to \operatorname{Coker}(\beta_i \otimes \beta'_i) \to \mathcal{F}_{i+1} \otimes \operatorname{Coker} \beta'_i \to 0.$$

An analogous sequence can be obtained for $\operatorname{Coker}(\alpha_i \otimes \alpha'_i)$. To prove (i), one may reduce the situation to the case where X is the spectrum of a complete local ring R. Because the support of $\operatorname{Coker}(\beta_i \otimes \beta'_i)$ is on $X \times \{\infty\}$, we may the replace $X \times C$ by the completion Y of $X \times C$ along $X \times \{\infty\}$, which is the spectrum of a semi-local ring. But then the $\mathcal{F}_i$ and $\mathcal{F}'_i$ are free on Y and the assertion (i) is obvious. A similar argument, in which one replaces $X \times C$ by its completion along the graph of the map char, proves the assertion (ii). ∎

In the same way in which we defined $\mathcal{F}_{\mathrm{DA}} \otimes \mathcal{F}'_{\mathrm{DA}}$ above, one can for a given pure Drinfeld-Anderson motive $\mathcal{F}_{\mathrm{DA}}$ of rank r define Drinfeld-Anderson data $\otimes^n \mathcal{F}_{\mathrm{DA}}$, $\mathrm{Sym}^n \mathcal{F}_{\mathrm{DA}}$ and $\bigwedge^n \mathcal{F}_{\mathrm{DA}}$ of ranks r^n, $\binom{n+r-1}{n}$ and $\binom{r}{n}$, respectively.

Proposition 9.6 *For a pure Drinfeld-Anderson motive $\mathcal{F}_{\mathrm{DA}}$ and any $n \in \mathbb{N}$, the Drinfeld-Anderson data $\otimes^n \mathcal{F}_{\mathrm{DA}}$, $\mathrm{Sym}^n \mathcal{F}_{\mathrm{DA}}$ and $\bigwedge^n \mathcal{F}_{\mathrm{DA}}$ define pure Drinfeld-Anderson motives of rank r^n, $\binom{n+r-1}{n}$ and $\binom{r}{n}$, respectively, each of weight nw.*

Proof: The assertion for $\otimes^n \mathcal{F}_{\mathrm{DA}}$ is an immediate consequence of the previous lemma. The other two Drinfeld-Anderson data are defined as natural quotients of $\otimes^n \mathcal{F}_{\mathrm{DA}}$. For these, we will check (i)–(iii) of the above definition. As in the proof of the lemma, part (iii) is easy and omitted.

Using the same reduction procedure as in the proof of the previous lemma, we may assume that $X = \mathrm{Spec}\, R$, where R is a complete local ring, and we may replace $X \times C$ by $\mathrm{Spec}\, S$ where $S = R[[x]]$. In this situation, the proof is completed by the following lemma. ∎

Lemma 9.7 *Let R be complete local with maximal ideal $\mathfrak{m}$ and $S = R[[x]]$. Suppose $\varphi \in M_{r\times r}(S)$ is such that one has a short exact sequence*

$$0 \longrightarrow S^r \xrightarrow{\varphi} S^r \longrightarrow C \longrightarrow 0, \tag{43}$$

in which C is a free R-module of rank d, supported on $\mathrm{Spec}\, S/(x)$. Then for all $n \in \mathbb{N}$ the map $\mathrm{Sym}^n \varphi$ is injective and its cokernel is free over R of rank $d \cdot \binom{n+r-1}{n}$. Similarly, for $n \in \{1, \ldots, n\}$, the map $\bigwedge^n \varphi$ is injective and its cokernel is free over R of rank $d \cdot \binom{r}{n}$.

Proof: We only give the proof for the symmetric powers, the other one being similar. Observe first that φ becomes an isomorphism, if we invert x and that the map $S \hookrightarrow S[x^{-1}]$ is injective. Since computing symmetric powers of φ commutes with inverting x, it follows that the map $\mathrm{Sym}^n \varphi : \mathrm{Sym}^n S^r \to \mathrm{Sym}^n S^r$ is injective, because after inverting x, it is an isomorphism.

Define C_n as the cokernel of $\mathrm{Sym}^n \varphi$. Set $\kappa := R/\mathfrak{m}$ and let $\bar{\varphi}$ be the reduction of φ modulo $\mathfrak{m}$. Clearly

$$0 \longrightarrow \mathrm{Sym}^n S^r \xrightarrow{\mathrm{Sym}^n \varphi} \mathrm{Sym}^n S^r \longrightarrow C_n \longrightarrow 0$$

is an R-flat resolution of C_n, so that $\mathrm{Tor}_1^R(C_n, \kappa)$ is isomorphic to the kernel of $\mathrm{Sym}^n \bar{\varphi}$. Because C is R-flat, reducing the short exact sequence (43) modulo $\mathfrak{m}$ yields a short exact sequence

$$0 \longrightarrow \kappa[[x]]^r \xrightarrow{\bar{\varphi}} \kappa[[x]]^r \longrightarrow \bar{C} \longrightarrow 0,$$

where $\bar{C}$ has dimension r as a vector space over κ. Over the valuation ring $\kappa[[x]]$ it is easy to see that $\mathrm{Sym}^n \bar{\varphi}$ is injective and that its cokernel has dimension $d \cdot \binom{n+r-1}{n}$ over κ (e.g., by using the elementary divisor theorem). The first observation shows that $\mathrm{Tor}_1^R(C_n, \kappa) = 0$, so that C_n is flat over the local ring R. The second observation implies that $\mathrm{rank}_R\, C_n = \dim_\kappa \mathrm{Coker}(\mathrm{Sym}^n \bar{\varphi}) = d \cdot \binom{n+r-1}{n}$. This completes the proof of the lemma. ∎

For later use, we record the following consequence of Propositions 9.4 and 9.6.

Corollary 9.8 *The τ-sheaves $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} = \mathrm{Sym}^n \underline{\mathcal{F}}_{\mathcal{K}}$ arise from pure Drinfeld-Anderson motives of rank $n+1$ and weight $n/2$ on $\mathfrak{M}_{\mathcal{K}}$. In particular, they are families of pure A-motives on $\mathfrak{M}_{\mathcal{K}}$.*

9.2 Uniformizability

In this subsection, we assume that all schemes X and affinoid spaces $\mathfrak{X}$ are reduced. Therefore the spectral norm $|\ |$ will always define a norm on $\mathcal{O}_{\mathfrak{X}}$.

We introduce an ad hoc definition for *analytic family of globally uniformizable τ-sheaves*, which follows closely Anderson's pointwise definition, [1], § 2. No attempt is made to obtain great generality. The central result is a criterion, similar to the pointwise one given in [1], for a τ-sheaf attached to an A-module over an analytic base to be globally uniformizable. This can then be applied to Drinfeld-A-modules. As a corollary, we find an 'explicit' isomorphism on the analytic site between $\underline{\tilde{\mathcal{F}}}_{\Omega_{\mathcal{K}}|\mathfrak{X}}$ and $\underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A} \otimes_A P_{\mathfrak{X}}$ where $\mathfrak{X} \subset \Omega_{\mathcal{K}}$ is any connected component and $P_{\mathfrak{X}}$ is a suitable projective rank 2 module over A. In particular this implies that for any affinoid $\mathfrak{U}$ in the standard cover of $\mathfrak{U}_{\mathcal{K}}$ of $\overline{\mathfrak{M}}_{\mathcal{K}}^{\mathrm{rig}}$ which does not contain a cusp, there is an isomorphism

$$\underline{\tilde{\mathcal{F}}}_{\mathcal{K}|\mathfrak{U}\times\mathfrak{D}_A} \cong \underline{\tilde{\mathbb{1}}}_{\mathfrak{U},\mathfrak{D}_A} \otimes_A P_{\mathfrak{U}}$$

in $\widetilde{\mathbf{Coh}}_{\tau}(\mathfrak{U},\mathfrak{D}_A)$ for some projective rank 2 module $P_{\mathfrak{U}}$ over A.

Let $\mathfrak{G}/\mathfrak{X}$ denote the category of (rigid) vector bundles over $\mathfrak{X}$ with k-linear algebraic morphisms $\alpha : \mathcal{V} \to \mathcal{W}$, i.e., rigid locally on $\mathrm{Spm}\,\mathcal{R}$ the morphism α can be given by a polynomial $\sum A_i\sigma^i : \mathcal{R}^m \to \mathcal{R}^n$, where $A_i \in M_{m,n}(\mathcal{R})$, the restrictions of $\mathcal{V}$ and $\mathcal{W}$ to $\mathrm{Spm}\,\mathcal{R}$ arise from $\mathcal{R}^m$, respectively $\mathcal{R}^n$, and σ denotes the q-power map on the components of $\mathcal{R}^m$. If $\mathcal{V}$ is such a vector bundle, then $\mathrm{End}(\mathrm{Lie}(\mathrm{End}_{\mathfrak{G}/\mathfrak{X}}(\mathcal{E}))) \cong \mathrm{End}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{E})$, and hence there is a diagonal action of $a \in A \subset K_\infty \subset L \subset \Gamma(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ on it.

Definition 9.9 *A family of* A-modules $\underline{\mathcal{E}}$ of dimension d and rank r on a rigid space $\mathfrak{X}/L$ *consists of a vector bundle $\mathcal{E}$ of rank d on $\mathfrak{X}$ and a ring homomorphism*

$$\varphi\colon A \to \mathrm{End}_{\mathfrak{G}/\mathfrak{X}}(\mathcal{E}) : a \mapsto \varphi_a$$

subject to the following conditions: Let us denote by $\mathrm{Lie}(\mathcal{E})$ the tangent space of the k-group scheme $\mathcal{E}$ along the zero section and $\varphi'_a \in \mathrm{End}(\mathrm{Lie}\,\mathcal{E})$ the induced derivation of φ_a. Then:

(i) $\forall a \in A\, \exists n_a \in \mathbb{N} :\ (a - \varphi'_a)^{n_a}\,\mathrm{Lie}(\mathcal{E}) = 0.$

(ii) As a sheaf of $A_{\mathfrak{X}}$-modules, $\mathcal{M}(\mathcal{E}) := \mathrm{Hom}_{\mathfrak{G}/\mathfrak{X}}(\mathcal{E},\mathbb{G}_a)$ is coherent, where $a \in A$ acts as composition on the right with φ_a.

(iii) The sheaf $\mathcal{M}(\mathcal{E})$ is a locally free sheaf of $A_{\mathfrak{X}}$-modules of rank r, i.e., there exists an admissible affinoid cover $\mathfrak{U}_i = \mathrm{Spec}\,\mathcal{R}_i$ of $\mathfrak{X}$ such that the restriction of $\mathcal{M}(\mathcal{E})$ to $\mathrm{Spec}\,\mathcal{R}_i \otimes A$ is isomorphic to $\mathcal{P}_i^{\mathrm{as}}$ for some projective $\mathcal{R}_i \otimes A$-module $\mathcal{P}_i$ of rank r.

We define an operation τ on $\mathcal{M}(\mathcal{E})$ as composition on the left with the Frobenius on $\mathbb{G}_{a/\mathfrak{X}}$. Hence the sheaf $\mathcal{M}(\mathcal{E})$ is equipped with the structure of a rigid τ-sheaf over A on $\mathfrak{X}$. By $\underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}})$, we denote the induced rigid τ-sheaf on $\mathfrak{X}$ over $\mathfrak{D}_A$.

Let $\underline{\mathcal{E}}$ be a family of A-modules on a rigid space $\mathfrak{X}$ and fix an affinoid subdomain $U := \mathrm{Spm}\,B$ of $\mathfrak{X}$ over which $\mathcal{E}$ is free, say of rank r. Let $\kappa\colon \Gamma(U,\mathcal{E}) \to B^d$ be a fixed isomorphism and κ' the induced isomorphism $\Gamma(U,\mathrm{Lie}(\mathcal{E})) \to B^d$. The isomorphisms κ and κ' are called *coordinates* of $\mathcal{E}_{|U}$ and $\mathrm{Lie}(\mathcal{E})_{|U}$, respectively. For a matrix $\xi := (x_{i,j})$ over B, one defines $|\xi| := \sup |x_{i,j}|$, $\mathrm{v}(A) := \min\{\mathrm{v}(x_{i,j})\}$ and $\xi^{(p)} := (x_{i,j}^p)$. As in [1], § 2.1, one can show:

For fixed coordinates κ, expressing φ'_a and φ_a with respect to the coordinates κ' and κ, respectively, yields $\varphi_a(z) = \sum_{i=0}^{d_a} f_i(a) z^{(q^i)}$ for some $d_a \in \mathbb{N}$, $f_i(a) \in M_d(B)$ and $\varphi'_a = f_0(a)$ such that $f_0(a) - a$ is nilpotent. As in [1], § 2.1, one can show the following:

Lemma 9.10 *There exists a unique sequence of matrices (e_n) in $M_d(B)$ with $e_0 = \mathrm{id}$, $\lim_{n\to\infty} \log q^{-n}|e_n| = -\infty$ such that for each $a \in A$, the rigid analytic function $\exp_{U,\mathcal{E}} : B^d \to B^d : z \mapsto \sum_{n\geq 0} e_n z^{(q^n)}$ satisfies*

$$\exp_{U,\mathcal{E}} \circ \varphi'_a = \varphi_a \circ \exp_{U,\mathcal{E}} .$$

The existence of such an exponential function as a formal power series can be obtained by an argument similar to the one given in [23], Sect. 4.6, however one has to work with vectors and matrices over B instead of scalars over a field. Because we assume that B is reduced, it is possible to reduce the global convergence proof to a pointwise computation - where one makes use of the fact that B is equipped with the supremum norm. The following sublemma on linear algebra allows it to make pointwise use of the proof in [1], § 2.1. Details of the proof of the lemma and the subsequent sublemma are left to the reader. It is in fact possible to prove the above lemma without the assumption that B is reduced. A full proof is given in forthcoming work by Hartl, cf. [25].

Sublemma 9.11 *Suppose F is a complete $\mathbb{R}$-valued field of positive characteristic with valuation $|\ | : F \to \mathbb{R}$. Let $e \in \mathrm{GL}_d(F)$ be a matrix such that $e - \mathrm{id}$ is nilpotent and let $M_d(F)$ be a normed vector space under $|(\alpha_{i,j})|' := \sup_{i,j} |\alpha_{i,j}|$. Then there exists $\alpha \in \mathrm{GL}_d(F)$ with $|\alpha|' = 1 = |\alpha^{-1}|$ and such that $e_1 := \alpha^{-1} e \alpha^{(q)}$ is upper triangular with ones on the diagonal.*

Furthermore, there exists $\beta \in \mathrm{GL}_d(F)$ with $|\beta|', |\beta^{-1}|' \leq (|e|')^d$ such that $e_2 := \beta^{-1} f \beta^{(q)}$ is again upper triangular with ones on the diagonal and satisfies $|e_2|' = |e_2^{-1}|' = 1$.

As the $\exp_{U,\mathcal{E}}$ in the previous lemma are unique, patching yields:

Proposition 9.12 *There exists a unique morphism $\exp_{\mathcal{E}} : \mathrm{Lie}(\mathcal{E}) \to \mathcal{E}$ of rigid spaces such that for all $\xi \in \mathrm{Lie}(\mathcal{E})$ one has $\exp_{\mathcal{E}}(\varphi'_a \xi) = \varphi_a(\exp_{\mathcal{E}} \xi)$ and such that on any affinoid $U = \mathrm{Spm}\, B$ of $\mathfrak{X}$ which trivializes $\mathcal{E}$, and for any coordinates of $\mathcal{E}_{|U}$, the morphism $\exp_{\mathcal{E}|U}$ takes the form given in the previous lemma.*

For various constructions below, we choose a non-constant $a \in A$ and a uniformizer $\pi \in K_\infty$ such that $a = \pi^{\mathrm{v}_\infty(a)}$. (To obtain such a pair (a, π) one can use the Riemann-Roch theorem.) Via a, we regard A as an algebra over $k[T]$ by mapping T to a, so that the map induces a finite, possibly ramified cover $C \to \mathbb{P}^1$. Let r' be the degree of this cover. The place ∞ of C is the only place above the infinite place of $\mathbb{P}^1$.

In the sequel, we will need the following technical lemma:

Lemma 9.13 *The action $\varphi' : A \to \mathrm{End}(\mathrm{Lie}\, \mathcal{E})$ extends uniquely to a continuous action of K_∞.*

PROOF: Because $K_\infty = A \otimes_{k[a]} k((1/a))$, we may assume for the proof that $A = k[T]$. Let θ be the image of T in K_∞. Furthermore, we may work locally, since any continuous extension will be unique. Write $\mathfrak{X} = \mathrm{Spm}\, B$ and

$\varphi'_T = \theta(I+N)$ with a nilpotent matrix $N \in M_d(B)$. For $n \in \mathbb{N}$, we define $\varphi'_{T^{-n}} := \theta^{-n} \sum_{i=0}^{r-1} \binom{-n}{i} N^i$. Then $|\varphi'_{T^{-n}}| \leq |\theta|^{-n} \cdot \max_{i=0,\ldots,r-1}\{|N^i|\}$, so that $\lim_{n\to\infty} |\varphi'_{T^{-n}}| = 0$. We define the action of $\sum_{n\ll\infty} a_n T^n$ on $\operatorname{Lie}\mathcal{E}$ as the endomorphism

$$\sum_{n\ll\infty} a_n \theta^n \sum_{i=0}^{r-1} \binom{-n}{i} N^i \in M_d(B).$$

The sum converges because $|\theta^{-1}| < 1$. Continuity of the endomorphism is easily established. For the uniqueness, observe that by continuity it suffices to show that any two extensions φ' and $\tilde{\varphi}'$ take the same values on $\{T^n : n \in \mathbb{Z}\}$. But since both are ring homomorphisms and since φ'_T is invertible, this assertion is obvious. ∎

It will also be useful to have locally a rigid analytic inverse of $\exp_{\mathcal{E}}$.

Lemma 9.14 *For each affinoid U which trivializes $\mathcal{E}$ and any coordinates κ, there exists a unique sequence of matrices (l_n) in $M_d(B)$ such that $\log_{U,\mathcal{E}} : z \mapsto \sum_{n\geq 0} l_n z^{(q^n)}$ converges on a neighborhood V of $0 \in B$ and such that*

$$\exp_{U,\mathcal{E}} \circ \log_{U,\mathcal{E}} = \mathrm{id} \quad \textit{on } V.$$

Proof: We first consider the formal equation

$$\sum_{n\geq 0} e_n z^{(q^n)} \circ \sum_{n\geq 0} l_n z^{(q^n)} = z.$$

Because $e_0 = \mathrm{id}$, it yields the recursive equations

$$l_0 = \mathrm{id}, \;\; l_n = -(e_1 l_{n-1}^{(q)} + e_2 l_{n-2}^{(q^2)} + \ldots + e_n l_0^{(q^n)}).$$

In particular, if $\log_{\mathcal{E}}$ exists as an analytic function, it will clearly be unique.

Define $v_n := \min_{m=0,\ldots,n}\{\mathrm{v}(l_m)/q^m\}$ and set $c := \min_{i\in\mathbb{N}_0} \mathrm{v}(e_i) \leq 0$. The above recursive formulas for the l_n imply that $v_{n+1} \geq v_n + c/q^n$, and hence that $v_n \geq cq/(q-1) =: c'$ for all $n \in \mathbb{N}_0$. Thus the formally defined series $\sum_{n\geq 0} l_n z^{(q^n)}$ converges for $v(z) > c'$, and the proof of the lemma is complete. ∎

Definition 9.15 *Define $H_*(\underline{\mathcal{E}})$ as the kernel of $\exp_{\mathcal{E}}$ in the vector bundle $\operatorname{Lie}\mathcal{E}$.*

Clearly there is an action of A on $H_*(\underline{\mathcal{E}})$ induced from φ' on $\operatorname{Lie}\mathcal{E}$.

Lemma 9.16 *For any connected affinoid subdomain U of $\mathfrak{X}$, the set of sections $\Gamma(U, H_*(\underline{\mathcal{E}}))$ is a projective A-module of rank not exceeding r.*

Proof: Let x be an L'-valued point in U for a finite extension L' of L, and let $\underline{\mathcal{E}}_x$ be the stalk of $\underline{\mathcal{E}}$ at x. This is an A-module in the sense of Anderson.

By the explicit inverse to $\exp_{\mathcal{E}}$ constructed in the previous lemma, it follows that $\inf_{x\in U} |s(x) - s'(x)| > 0$ for any distinct sections $s, s' \in \Gamma(U, H_*(\underline{\mathcal{E}}))$. Combining this with the connectedness of U yields that the map

$$\Gamma(U, H_*(\underline{\mathcal{E}})) \to H_*(\underline{\mathcal{E}}_x) : s \mapsto s(x)$$

is an injective map of A-modules. By [1], Lem. 2.4.1, the module $H_*(\underline{\mathcal{E}}_x)$ is free over $k[T]$ of rank at most $r'r$, and hence projective over A of rank at most r. ∎

Note that in the course of the proof of Lemma 9.16, we showed that for connected affinoid U the specialization map $\Gamma(U, H_*(\underline{\mathcal{E}})) \to H_*(\underline{\mathcal{E}}_x)$ is injective for all points $x \in U$.

Definition 9.17 *A family of A-modules $\underline{\mathcal{E}}$ of rank r is* globally uniformizable *on $\mathfrak{X}$ if $\Gamma(\mathfrak{X}, H_*(\underline{\mathcal{E}}))$ is a projective A-module of rank r and if for all $x \in \mathfrak{X}$ the map*

$$\Gamma(\mathfrak{X}, H_*(\underline{\mathcal{E}})) \to H_*(\underline{\mathcal{E}}_x)$$

is an isomorphism.

A rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{X}$ over $\mathfrak{D}_A$ is called trivial *if on each connected component $\mathfrak{X}'$ of $\mathfrak{X}$, there exists a projective finitely generated A-module $P_{\mathfrak{X}'}$ such that $\underline{\tilde{\mathcal{F}}}_{|\mathfrak{X}'\times\mathfrak{D}_A} \cong \underline{\tilde{\mathbb{1}}}_{\mathfrak{X}',\mathfrak{D}_A} \otimes_A P_{\mathfrak{X}'}$.*

A rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\mathfrak{X}$ over A is called globally uniformizable *if $\underline{\tilde{\mathcal{F}}}^{\mathfrak{D}_A\text{-rig}}$ is trivial. A rigid crystal is called* globally uniformizable*, if it is representable by a globally uniformizable rigid τ-sheaf.*

If $\mathfrak{X}$ is a finite set of points, then we simply speak of uniformizability instead of global uniformizability.

Remark 9.18 In [1], uniformizability is defined be requiring the exponential map to be surjective. It is easy to see that this is only pointwise a good definition. We therefore used an equivalent, more appropriate condition in our definition of global uniformizability.

The following is the main result of this subsection.

Theorem 9.19 *For a family of A-modules $\underline{\mathcal{E}}$ on $\mathfrak{X}$ of rank r the following are equivalent:*

(i) $\underline{\mathcal{E}}$ is globally uniformizable.

(ii) The family $\underline{\mathcal{M}}(\underline{\mathcal{E}})$ of rigid A-motives on $\mathfrak{X}$ is globally uniformizable.

(iii) $\underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}})$ is trivial.

Before we come to the proof, we state two corollaries. Fix an admissible $\mathcal{K} \subset \mathrm{GL}_2(\mathbb{A}^f)$. Recall that on page 44, we defined a Drinfeld-module $(\underline{\varphi}(\mathcal{K}), \psi(\mathcal{K}))$ with a level $\mathcal{K}$-structure on $\Omega_{\mathcal{K}}$ from the local system $(\mathcal{L}, \Lambda, s, \psi)$. In particular, $\underline{\varphi}(\mathcal{K})$ is by its very definition globally uniformizable on $\Omega_{\mathcal{K}}$. The lattice on Ω_g, $g \in \mathrm{GL}_2(\mathbb{A}^f)$, which defines $\underline{\varphi}(\mathcal{K})$, is given by $\Lambda_g = \hat{A}^2 g^{-1} \cap K^2$. Recall that we defined $\underline{\tilde{\mathcal{F}}}(\mathcal{K})$ as the rigid family of A-motives on $\Omega_{\mathcal{K}}$ corresponding to $\underline{\varphi}(\mathcal{K})$.

Corollary 9.20 *The rigid τ-sheaf $\underline{\tilde{\mathcal{F}}}(\mathcal{K})^{\mathfrak{D}_A\text{-rig}}$ is trivial on $\Omega_{\mathcal{K}}$. On the connected component $\Omega_g \subset \Omega_{\mathcal{K}}$ it is isomorphic to $\underline{\tilde{\mathbb{1}}}_{\Omega_g,\mathfrak{D}_A} \otimes_A \mathrm{Hom}_A(\Lambda_g, \Omega_A)$.*

The proof is given at the end of this section after that of Theorem 9.19.

Corollary 9.21 *Let $\overline{\mathfrak{U}}_{\mathcal{K}}$ be the standard cover of $\overline{\mathfrak{M}}_{\mathcal{K}}^{\mathrm{rig}}$ for an admissible $\mathcal{K}$. Assume that $\mathfrak{U} \in \overline{\mathfrak{U}}_{\mathcal{K}}$ does not contain a cusp. Then $\underline{\mathcal{F}}_{\mathcal{K}}^{\mathfrak{D}_A\text{-rig}}\Big|_{\mathfrak{U}}$ is globally uniformizable.*

Proof: An affinoid $\mathfrak{U} \in \overline{\mathfrak{U}}_{\mathcal{K}}$ which does not contain a cusp corresponds to a stable simplex t of the Bruhat-Tits tree of $\mathrm{GL}_2(A)$. The stability of t implies that there is an isomorphism between an affinoid $\mathfrak{V}$ of $\Omega_{\mathcal{K}}$ and $\mathfrak{U}$ in such a way that $\underline{\mathcal{F}}_{\mathcal{K}}^{\mathfrak{D}_A\text{-rig}}{}_{|\mathfrak{U}\times\mathfrak{D}_A}$ is isomorphic to $\underline{\tilde{\mathcal{F}}}(\mathcal{K})^{\mathfrak{D}_A\text{-rig}}_{|\mathfrak{V}\times\mathfrak{D}_A}$. But the latter is globally uniformizable by the previous corollary. ∎

We now start with the preparations for the proof of Theorem 9.19, which is quite similar to [1], § 2.

Definition 9.22 *A* topological A-module $\mathbb{T}$ *is a commutative topological group equipped with an A-module structure in such a way that for all $a \in A$ the map $f \mapsto af : \mathbb{T} \to \mathbb{T}$ is continuous.*

Given topological A-modules $\mathbb{T}, \mathbb{T}'$, let $\mathrm{Hom}^c_A(\mathbb{T}, \mathbb{T}')$ *denote the group of continuous homomorphisms $\mathbb{T} \to \mathbb{T}'$ which are compatible with the A-multiplication.*

Let $\underline{\mathcal{E}}$ be as above and $U = \mathrm{Spm}\, B$ a trivialization with coordinates κ. Then (B^d, φ) is a topological A-module. Furthermore the sequence

$$0 \longrightarrow A \longrightarrow K_\infty \longrightarrow K_\infty/A \longrightarrow 0 \tag{44}$$

is a short exact sequence of topological A-modules.

Proposition 9.23 *Applying the functor* $\mathrm{Hom}^c_A(\_, (B^d, \varphi))$ *to the short sequence (44) yields a left exact sequence which is isomorphic to*

$$0 \longrightarrow \Gamma(U, H_*(\underline{\mathcal{E}})) \longrightarrow (B^d, \varphi') \overset{\exp_{U,\mathcal{E}}}{\longrightarrow} (B^d, \varphi). \tag{45}$$

PROOF: Clearly $\mathrm{Hom}^c_A(A, (B^d, \varphi)) \cong (B^d, \varphi)$. We claim that the map

$$\mathrm{Lie}\, \mathcal{E} \to \mathrm{Hom}^c_A(K_\infty, (B^d, \varphi)) : x \mapsto (\psi_x : y \mapsto \exp_{\mathcal{E}}(xy)) \tag{46}$$

is an isomorphism. The claim easily implies the proposition, and so we will now prove it.

The well-definedness of the map given follows from Lemma 9.13. Continuity of ψ_x follows from the continuity of $\exp_{U,\mathcal{E}}$ and that of the action of K_∞ on $\mathrm{Lie}\, \mathcal{E}$. That each ψ_x is an A-module map is a consequence of the defining property of $\exp_{\mathcal{E}}$. To prove injectivity, we assume $\psi_x = \psi_{x'}$ for $x, x' \in B$. It follows that $\exp_{U,\mathcal{E}}((x - x')y) = 0$ for all $y \in K_\infty$. Choosing y sufficiently close to zero, but different from zero, we may assume that $|(x - x')y| < c$, where c is the radius of convergence of $\log_{U,\mathcal{E}}$. Because $\exp_{\mathcal{E}}$ has $\log_{U,\mathcal{E}}$ as a local inverse, we must have $(x - x')y = 0$, and hence $x = x'$ because $y \in K_\infty \smallsetminus \{0\}$.

Finally to prove surjectivity, let ψ be an element in $\mathrm{Hom}^c_A(K_\infty, (B^d, \varphi))$, and define $w_n := \psi(a^{-n})$, where a is defined above Lemma 9.13. Because the a^{-n} tend to zero on U, there exists an n_0 such that $|w_n| < c$ for $n \geq n_0$. Define $x_n := \log_{U,\mathcal{E}}(w_n)$ for $n \geq n_0$. It follows that $\varphi'_{a^m} x_n = x_{n-m}$ for $n, m \in \mathbb{N}$ such that $n - m \geq n_0$. Defining $x := \varphi'_{a^n} x_n$ for $n \geq n_0$, one has the equality $\psi_x = \psi$ on the sequence $\{a^{-n} : n \geq n_0\}$. Hence by continuity and the homomorphism property of ψ and ψ_x, the two morphisms must agree. ∎

Definition 9.24 *Given a topological A-module $\mathbb{T}$, we denote by* $\mathrm{Hom}^c(\mathbb{T}, B)$ *the group of continuous homomorphisms $f : \mathbb{T} \to \mathbb{G}_a(B)$ equipped with the unique structure of left $B\{\tau\} \otimes A$-module for which*

$$(bf)(c) = b(f(c)),\ \ (af)(c) = f(ac),\ \ (\tau f)(c) = f(c)^q,$$

for all $b \in B$, $f \in \mathrm{Hom}^c(\mathbb{T}, B)$ and $c \in \mathbb{T}$.

Applying the functor $\mathrm{Hom}^c(\_, B)$ to the exact sequence (44), which is split exact as a sequence of topological groups, yields an exact sequence

$$0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow Z_3 \longrightarrow 0$$

of left $B\{\tau\} \otimes A$-modules, where

$$\begin{aligned} Z_1 &:= \operatorname{Hom}^c(K_\infty/A, B), \\ Z_2 &:= \operatorname{Hom}^c(K_\infty, B), \\ Z_3 &:= \operatorname{Hom}^c(A, B). \end{aligned}$$

We omit the proof of the following result which is straightforward, if tedious.

Lemma 9.25 *For any of the topological A-modules $\mathbb{T}$ in the sequence (44), the map*

$$\operatorname{Hom}^c_A(\mathbb{T}, (B^d, \varphi)) \longrightarrow \operatorname{Hom}_{B\{\tau\}\otimes A}(\Gamma(U, \mathcal{M}(\mathcal{E})), \operatorname{Hom}^c(\mathbb{T}, B)) :$$
$$f \mapsto (m \mapsto (c \mapsto m(f(c))))$$

is an isomorphism.

Abbreviate $B_{\mathfrak{D}_A} := B\hat{\otimes}A\langle\langle T\rangle\rangle := B\hat{\otimes}_L(A \otimes_{k[T]} L\langle\langle T\rangle\rangle)$ and define $B_{\mathfrak{D}_A}\{\tau\}$ as the set of finite formal sums $B\{\tau\}\hat{\otimes}A\langle\langle T\rangle\rangle$ equipped with the addition law

$$\Big(\sum_j f_j\tau^j\Big) + \Big(\sum_j g_j\tau^j\Big) = \Big(\sum_j (f_j+g_j)\tau^j\Big)$$

and the multiplication

$$\Big(\sum_j f_j\tau^j\Big)\Big(\sum_{j'} g_{j'}\tau^{j'}\Big) = \Big(\sum_j\sum_{j'} (f_j g_{j'}^{(q^j)})\tau^{j+j'}\Big),$$

where the $f_j, g_{j'}$ are in $B\hat{\otimes}A\langle\langle T\rangle\rangle$.

Let $a_1, \ldots, a_s$ be a $k[T]$-basis of A.

Lemma 9.26 *The $B\{\tau\}\otimes A$-module Z_1 carries a unique structure of $B_{\mathfrak{D}_A}\{\tau\}$-module such that*

$$(fh)(c) = \sum_{i,j} b_{i,j} \otimes a_i h(T^j c)$$

for all $h \in Z_1$, $c \in K_\infty/A$ and $f = \sum_{i\geq 0}\sum_{j=1}^s b_{i,j}\otimes a_j T^i \in B\hat{\otimes}A\langle\langle T\rangle\rangle$.

Proof: The expressions $h(a_jT^ic)$ are well-defined for $i \in \mathbb{Z}$, $j = 1, \ldots, r$ by Lemma 9.13. Because K_∞/A is compact and h is continuous, the image of h is bounded. As $b_{i,j} \to 0$ for $i \to \infty$, the series $\sum_{i,j} b_{i,j}h(a_jT^ic)$ converges to an element in B. The well-definedness of the action of $B_{\mathfrak{D}_A}\{\tau\}$ is now clear, and we leave it to the reader to check that this defines the structure of a $B_{\mathfrak{D}_A}\{\tau\}$-module on Z_1. ∎

Let $\operatorname{Res}_\infty : \Omega_{K_\infty} \to k_\infty$ denote the residue map at ∞. A *topological k-basis* of K_∞/A is defined as a countable sequence of elements $(a_i)_{i\in I} \subset K_\infty/A$ converging to zero such that every element in K_∞/A can be written uniquely as a converging sum $\sum_{i,j}\alpha_i a_i$ with $\alpha_i \in k$.

Lemma 9.27 *The bilinear map*

$$\Omega_A \times K_\infty \longrightarrow k_\infty : (\omega, b) \mapsto \operatorname{Res}_\infty(b\omega)$$

vanishes on $\Omega_A \times A$ and hence induces a map

$$\langle \_\_, \_\_\rangle \colon \Omega_A \times K_\infty/A \longrightarrow k_\infty : (\omega, b) \mapsto \operatorname{Res}_\infty(b\omega).$$

Let furthermore $\omega_1, \ldots, \omega_r$ be a basis of Ω_A over $k[T]$. Then there exists a uniquely determined topological k-basis $\{a_{i,j}\}_{i\in\mathbb{N}_0, j\in\{1,\ldots,r\}}$ of K_∞/A determined by the condition

$$\langle T^{i'}\omega_{j'}, a_{i,j}\rangle = \delta_{i,i'}\delta_{j,j'}.$$

PROOF: For $\omega \in \Omega_K$ and $a \in K$ one has

$$\sum_{x \in C} \operatorname{Res}_x(a\omega) = 0,$$

where Res_x denotes the residue map at x. If ω lies in Ω_A and a in A, then all the residues except for the one at ∞ are zero. Hence the above formula implies $\operatorname{Res}_\infty(a\omega) = 0$, as asserted.

For the second part of the lemma, note that $\Omega_A = \varinjlim_{n\to\infty} H^0(C, \Omega_C(n\infty))$, and by Serre duality we have

$$H^0(C, \Omega_C(n\infty))^* \cong H^1(C, \mathcal{O}_C(-n\infty)). \tag{47}$$

Let us fix $m \gg 0$ and consider the cohomology sequence induced by

$$0 \longrightarrow \mathcal{O}_C(-n\infty) \longrightarrow \mathcal{O}_C(m\infty) \longrightarrow \mathcal{O}_C(m\infty)/\mathcal{O}_C(-n\infty) =: \mathcal{F}_{m,n} \longrightarrow 0$$

for $n \gg 0$:

$$\begin{tikzcd}
0 \arrow[r] & H^0(C, \mathcal{O}_C(m\infty)) \arrow[r] \arrow[d, "\cong"] & H^0(C, \mathcal{F}_{m,n}) \arrow[r] \arrow[d, "\cong"] & H^1(C, \mathcal{O}_C(-n\infty)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \{a \in A : \deg a \leq m\} \arrow[r] & \pi^{-m}A_\infty/\pi^n A_\infty \arrow[r] & H^1(C, \mathcal{O}_C(-n\infty)) \arrow[r] & 0
\end{tikzcd}$$

If we first pass to the inverse limit, as $n \to \infty$, and then to the direct limit, as $m \to \infty$, we obtain the short exact sequence

$$0 \longrightarrow A \longrightarrow K_\infty \longrightarrow \varprojlim_{n\to\infty} H^1(C, \mathcal{O}_C(-n\infty)) \longrightarrow 0.$$

So via Serre duality, the complete dual of Ω_A is identified with K_∞/A. One can make this explicit using repartitions as in [49]. Arguing in a similar way as in the first paragraph, one finds that this duality is simply given by $\operatorname{Res}_\infty$. The existence and uniqueness of a topological k-basis of K_∞/A with the desired properties can now be obtained by studying the inverse system which arises from (47) and by using the fact that the duality is given by $\operatorname{Res}_\infty$. This is simple and left to the reader. ∎

Lemma 9.28 *The map*

$$\begin{array}{ccc}
(B \otimes \Omega_A)\hat{\otimes}_L L\langle\langle T\rangle\rangle & \longrightarrow & Z_1 \\
\sum_{i,j} b_{i,j} \otimes \omega_j T^i & \mapsto & \Big(c \mapsto \sum_{i,j} b_{i,j} \operatorname{Res}_\infty(\omega_j T^i c)\Big),
\end{array}$$

where $b_{i,j} \to 0$ for $i \to \infty$, defines an isomorphism of $B_{\mathfrak{D}_A}\{\tau\}$-modules, where the τ-action on $(B \otimes \Omega_A)\hat{\otimes} L\langle\langle T\rangle\rangle$ is defined by $\tau(b \otimes \omega_j T^i) = b^q \otimes \omega_j T^i$.

PROOF: Because each element of $(B \otimes \Omega_A)\hat{\otimes} L\langle\langle T\rangle\rangle$ is uniquely represented as a sum $\sum_{i,j} b_{i,j} \otimes \omega_j T^i$, the well-definedness of the above map is clear. Furthermore it is easy to see that it defines an injective morphism of $B_{\mathfrak{D}_A}\{\tau\}$-modules.

To show surjectivity, given $f \in Z_1 = \operatorname{Hom}^c(K_\infty/A, B)$ we define elements $b_{i,j} := f(a_{i,j})$, where the $a_{i,j}$ form a topological k-basis of K_∞/A as constructed in the previous lemma. Since the $a_{i,j}$ converge to zero for $i \to \infty$, so do the $b_{i,j}$, and we may define

$$g := \sum_{i,j} b_{i,j} \otimes \omega_j T^i \in (B \otimes \Omega_A)\hat{\otimes} L\langle\langle T\rangle\rangle.$$

Clearly the image of g in Z_1 maps $a_{i,j}$ to $b_{i,j}$, and because the $a_{i,j}$ form a topological k-basis, it follows that the image of g agrees with f. ∎

We have $\underline{\tilde{\mathbb{1}}}_{B,\mathfrak{D}_A} \otimes_A \Omega_A \cong (B \otimes \Omega_A)\hat{\otimes} L\langle\langle T\rangle\rangle$, so that by the previous lemma and patching the following result is immediate:

Corollary 9.29 *For connected $\mathfrak{X}$, there are natural isomorphisms*

$$\begin{aligned}\Gamma(\mathfrak{X}, H_*(\underline{\mathcal{E}})) &\cong \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},A)}(\underline{\mathcal{M}}(\mathcal{E}), \underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A} \otimes_A \Omega_A) \\ &\cong \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},\mathfrak{D}_A)}(\underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}}), \underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A} \otimes_A \Omega_A).\end{aligned}$$

We now give the proof of the main result:

PROOF of Theorem 9.19: For the proof, we may assume that $\mathfrak{X}$ is connected. As the equivalence of (ii) and (iii) is obvious from the definitions, we only prove (i)$\Leftrightarrow$(iii). We first prove (iii)$\Rightarrow$(i). So let us assume that $\underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}})$ is trivial, so that $\underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}}) \cong \underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A} \otimes_A P$. Because $\underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}})$ is of rank r, it follows that P is of rank r over A. We display the situation in the following diagram, where the vertical maps are specialization maps:

$$\begin{array}{ccccc}\Gamma(\mathfrak{X}, H_*(\underline{\mathcal{E}})) & \xrightarrow[\text{Cor. 9.29}]{\cong} & \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},\mathfrak{D}_A)}(\underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A}\otimes_A P, \underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A}\otimes_A\Omega_A) & \xrightarrow{\cong} & \mathrm{Hom}_A(P,\Omega_A) \\ \downarrow & & \downarrow & & \downarrow \\ H_*(\underline{\mathcal{E}}_x) & \xrightarrow[\text{Cor. 9.29}]{\cong} & \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\{x\},\mathfrak{D}_A)}(\underline{\tilde{\mathbb{1}}}_{\{x\},\mathfrak{D}_A}\otimes_A P, \underline{\tilde{\mathbb{1}}}_{\{x\},\mathfrak{D}_A}\otimes_A\Omega_A) & \xrightarrow{\cong} & \mathrm{Hom}_A(P,\Omega_A).\end{array}$$

Since the right vertical map is clearly an isomorphism, it follows that $\underline{\mathcal{E}}$ is globally uniformizable on $\mathfrak{X}$.

For the converse (i)$\Rightarrow$(iii), let $\tilde{P} := \Gamma(\mathfrak{X}, H_*(\mathcal{E}))$ and define

$$\underline{b} \colon \underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}}) \longrightarrow \mathrm{Hom}_A(\tilde{P}, \underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A} \otimes_A \Omega_A) \cong \underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A} \otimes_A \mathrm{Hom}_A(\tilde{P}, \Omega_A)$$

by mapping $(m, f) \in \underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}}) \times \tilde{P}$ to $f(m) \in \underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A} \otimes_A \Omega_A$. Let us regard the map $\underline{b}$ over $k[T]$ where $k[T] \to A$ is a flat morphism of degree r'. Using our assumption that the $H_*(\underline{\mathcal{E}}_x)$ are of projective of rank r over A, and hence projective of rank rr' over $k[T]$, [1], 2.11, shows that the fibers $\underline{b}_x$ of $\underline{b}$ are isomorphisms for any x. Because $\mathfrak{X} \times \mathfrak{D}_{k[T]}$ is a reduced rigid space, it follows that $\underline{b}$ itself is an isomorphism. Therefore $\underline{\tilde{\mathcal{M}}}(\underline{\mathcal{E}})$ is isomorphic to $\underline{\tilde{\mathbb{1}}}_{\mathfrak{X},\mathfrak{D}_A} \otimes_A \mathrm{Hom}_A(\tilde{P}, \Omega_A)$ and the proof of the proposition is complete. ∎

PROOF of of Corollary 9.20: To prove the corollary, it will suffice to prove the second assertion. By its very construction, the Drinfeld-A-module $\underline{\varphi}(\mathcal{K})$ is a globally uniformizable A-motive of rank 2 in the sense of Definition 9.17. Thus by the above theorem, we may identify the restriction of $\underline{\mathcal{F}}(\mathcal{K})^{\mathfrak{D}_A\text{-rig}}$ to the connected rigid space Ω_g with $\underline{\tilde{\mathbb{1}}}_{\Omega_g,\mathfrak{D}_A} \otimes_A P_g$ for some projective A-module P_g. We have the following chain of isomorphisms:

$$\begin{aligned}\Lambda_g &\overset{\text{Def. of}\underline{\varphi}(\mathcal{K})}{\cong} H_*(\Omega_g, \underline{\varphi}(\mathcal{K})_{|\Omega_g}) \\ &\overset{\text{Cor. 9.29}}{\cong} \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Omega_g,\mathfrak{D}_A)}(\underline{\tilde{\mathbb{1}}}_{\Omega_g,\mathfrak{D}_A} \otimes_A P_g, \underline{\tilde{\mathbb{1}}}_{\Omega_g,\mathfrak{D}_A} \otimes_A \Omega_A) \\ &\cong \mathrm{Hom}_A(P_g, \Omega_A).\end{aligned}$$

This easily yields $P_g \cong \mathrm{Hom}_A(\Lambda_g, \Omega_A)$, as asserted. ∎

10 An Eichler-Shimura isomorphism

In this section we fix an admissible subgroup $\mathcal{K}$ of $\mathrm{GL}_2(\hat{A})$ of minimal conductor $\mathfrak{n}$. We also fix a set of representatives $\{t_\nu\}$ of $\mathrm{Cl}_\mathcal{K}$ and let x_ν be the corresponding elements in $\mathrm{GL}_2(\mathbb{A}^f)$ as in (14). By admissibility of $\mathcal{K}$, the corresponding arithmetic groups Γ_ν will all be p'-torsion free. Recall that $\mathfrak{b}^*_{K_\infty} : \operatorname{Spec} K_\infty \to \operatorname{Spec} A(\mathfrak{n})$ is the base change morphism and $g_\mathcal{K} : \mathcal{M}_\mathcal{K} \to \operatorname{Spec} A(\mathfrak{n})$ the structure morphism. Also, we recall that in Section 9 we defined $\underline{\mathcal{F}}_\mathcal{K}^{(n)} = \operatorname{Sym}^n \underline{\mathcal{F}}_\mathcal{K}$. Moreover, we use the convention that for a τ-sheaf or crystal $\underline{\mathcal{F}}$ on $\mathfrak{M}_\mathcal{K}$, considered as a scheme over $\operatorname{Spec} A(\mathfrak{n})$, we denote by $\underline{\mathcal{F}}^{\mathfrak{D}_A\text{-rig}}$ the object $(\mathfrak{b}^*_{K_\infty}\underline{\mathcal{F}})^{\mathfrak{D}_A\text{-rig}}$.

The basic idea is that the τ-sheaf $\underline{\mathcal{F}}_\mathcal{K}$ plays a similar role for Drinfeld-modular forms, as does $R^1_{\text{ét}} f_{\tilde{\Gamma}} \mathbb{Q}_l$ for classical modular forms, where $f_{\tilde{\Gamma}}$ is the map from the universal elliptic curve with level-$\tilde{\Gamma}$ structure to the corresponding moduli space. For example, for any proper non-zero ideal $\mathfrak{n}$ of A, the étale sheaf $(\underline{\mathcal{F}}_\mathcal{K} \otimes_A A/\mathfrak{n})_{\text{ét}}$ describes the space of $\mathfrak{n}$-torsion points as a Galois module in precisely the same way as $R^1_{\text{ét}} f_{\tilde{\Gamma}} \mathbb{Z}/(l^m)$ describes the Galois module of l^m-torsion points on the universal elliptic curve of level $\tilde{\Gamma}$. However $\underline{\mathcal{F}}_\mathcal{K}$ is a global and not a formal object, and so it carries indeed more information than the étale sheaf $R^1_{\text{ét}} f_{\tilde{\Gamma}} \mathbb{Q}_l$.

Carrying on with this analogy, it is natural to consider the following object:

Definition 10.1 *The* A-crystal of Drinfeld cusp forms on $\operatorname{Spec} A(\mathfrak{n})$ of weight $n+2$ and level $\mathcal{K}$ *is defined as*

$$\underline{\mathcal{S}}_{n+2}(\mathcal{K}) := R^1 g_{\mathcal{K}!} \underline{\mathcal{F}}_\mathcal{K}^{(n)}.$$

We define the rigid $\mathfrak{D}_A$-crystal on $\operatorname{Spm} K_\infty$ of Drinfeld cusp forms of weight $n+2$ and level $\mathcal{K}$ *as*

$$\underline{\mathcal{S}}_{n+2}^{\mathfrak{D}_A\text{-rig}}(\mathcal{K}) := \underline{\mathcal{S}}_{n+2}(\mathcal{K})^{\mathfrak{D}_A\text{-rig}}.$$

Let v be a place of K. We define the constructible étale v-adic sheaf on $\operatorname{Spec} A(\mathfrak{n})$ of Drinfeld cusp forms of weight $n+2$ and level $\mathcal{K}$ *as*

$$\underline{\mathcal{S}}_{n+2}^{\text{ét},v}(\mathcal{K}) := \varprojlim_m (\underline{\mathcal{S}}_{n+2}(\mathcal{K})/\mathfrak{p}_v^m \underline{\mathcal{S}}_{n+2}(\mathcal{K}))_{\text{ét}}.$$

Because extension by zero and commutes with pullbacks, taking symmetric powers and analytification, we have:

Proposition 10.2 $\underline{\mathcal{S}}_{n+2}^{\mathfrak{D}_A\text{-rig}}(\mathcal{K}) \cong H^1(\overline{\mathfrak{M}}^{\text{rig}}_{\mathcal{K},K_\infty}, j_{\mathcal{K}!} \operatorname{Sym}^n(\underline{\mathcal{F}}_\mathcal{K}^{\mathfrak{D}_A\text{-rig}}))$.

Note also that by Theorem 7.18, there is a canonical isomorphism

$$\underline{\mathcal{S}}_{n+2}^{\text{ét},v}(\mathcal{K}) \cong R^1_{\text{ét}} g_{\mathfrak{n}!}(\operatorname{Sym}^n \underline{\mathcal{F}}_\mathcal{K}^{\text{ét},v}).$$

Recall that in Proposition 7.5, we defined for a crystal $\underline{\mathcal{F}}$ over A the A-module $\underline{\mathcal{F}}^\tau$ as the set of global τ-invariant sections of any representing τ-sheaf. The main theorem of this section is the following:

Theorem 10.3 *For each admissible $\mathcal{K}$, there is an isomorphism*

$$\left(\mathbf{C}_n^{\text{St}}(\mathcal{K}, A)\right)^* \cong \left(\underline{\mathcal{S}}_n^{\mathfrak{D}_A-\text{rig}}(\mathcal{K})\right)^\tau.$$

This is called the *Eichler-Shimura isomorphism for Drinfeld cusp forms.*

10.1 The Eichler-Shimura map

To define the Eichler-Shimura map, we first give an explicit description of the complex $\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},\bullet}, \bar{M})$, for a local system $\bar{M}$ of left modules for $GL_2(\mathbb{A}^f)$. Then we compute the cohomology module $\underline{\mathcal{S}}_n^{\mathfrak{D}_A\text{-rig}}(\mathcal{K})$ via Čech cohomology for the standard affinoid cover of $\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K}}$ and show that the complex obtained by taking τ-invariants is directly isomorphic to $\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},\bullet}, V_n(\Lambda_g))$. This easily yields an injective morphism

$$\left(\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)\right)^* \longrightarrow \left(\underline{\mathcal{S}}_n^{\mathfrak{D}_A\text{-rig}}(\mathcal{K})\right)^\tau,$$

the desired Eichler-Shimura map. It being an isomorphism will result from analyzing $\mathfrak{n}$-torsion sheaves, and will be the subject of the following subsection.

Let $\bar{M} = (M, M_g)$ be any local system of left modules for $\mathrm{GL}_2(\mathbb{A}^f)$. For an explicit computation of $\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},\bullet}, \bar{M})$, we make the following choices: Let $R_{0,\nu}$ and $R_{1,\nu}$ denote sets of representatives stable vertices and edges of the tree $\mathcal{T} \times \{x_\nu \mathcal{K}\}$ with respect to Γ_ν, cf. page 64. Furthermore we define $R^o_{1,\nu}$ as the set of all oriented edges e such that the associated non-oriented edge $\bar{e}$ lies in $R_{1,\nu}$. For each oriented edge $e \in R^o_{1,\nu}$ with target $t(e)$, we have a unique vertex $v_e \in R_{0,\nu}$ and a unique $\gamma_e \in \Gamma_\nu$ such that

$$t(e) = \gamma_e v_e.$$

Giving a function $f_i \in \mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},i}, \bar{M})$ is thus equivalent to

(a) giving $f_0([v])$ for all $v \in \bigcup_\nu R_{0,\nu}$, and $f_0([\bar{e}])$ for all $\bar{e} \in \bigcup_\nu R_{1,\nu}$, for $i = 0$, and

(b) giving $f_1([e])$ for all $e \in \bigcup_\nu R^o_{1,\nu}$, for $i = 1$.

This yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},0}, \bar{M}) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},1}, \bar{M}) \arrow[d, "\cong"] \\
\bigoplus_\nu \left(\bigoplus_{v\in R_{0,\nu}} M_{x_\nu} \oplus \bigoplus_{v\in R_{1,\nu}} M_{x_\nu}\right) \arrow[r] & \bigoplus_\nu \left(\bigoplus_{e\in R^e_{1,\nu}} M_{x_\nu}\right).
\end{tikzcd}$$

If f_1 is the image of f_0 under the boundary map, then

$$f_1([e]) = f_0([\bar{e}]) - f_0([t(e)]) = f_0([\bar{e}]) - \gamma_e f_0([v_e]).$$

We recall that $[v_e]$ is zero if v_e is unstable.

Let us now compute the Čech complex for the standard cover $\bar{\mathfrak{U}}_{\mathcal{K}}$ of $\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K}}$ and the crystal $j_{\mathcal{K}!}\underline{\mathcal{F}}^{(n),\mathfrak{D}_A\text{-rig}}$, which computes the pushforward under $R^i g_{\mathcal{K}!}$ of $\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}}$ over the base $\mathrm{Spm}\, K_\infty$. This cover, which was constructed in Proposition 3.32 and Definition 4.17, can be described as follows:

(i) For each $\bar{e} \in \bigcup_\nu R_{1,\nu}$ one has an affinoid $\mathfrak{U}_{\bar{e}}$,

(ii) For each $v \in \bigcup_\nu R_{0,\nu}$ one has an affinoid $\mathfrak{U}_v$,

(iii) For each cusp c of $\mathrm{GL}_2(K)\backslash\Omega_{\mathcal{K}}$ one has an affinoid $\mathfrak{U}_c$.

Let us choose an order on the elements of $\bar{\mathfrak{U}}_{\mathcal{K}}$ with respect to which all the $\mathfrak{U}_v$ and all the $\mathfrak{U}_c$ are smaller than the $\mathfrak{U}_{\bar{e}}$. Let I be the union of all sets $R_{0,\nu}$, $R_{1,\nu}$ and the set of cusps. Then we have $\mathfrak{U}_i \cap \mathfrak{U}_{i'} = \varnothing$ for elements $i < i'$ of I, unless i is a vertex or a cusp and i' is an adjacent edge. If the intersection is non-empty, it is an annulus. Furthermore in this case we define e to be the edge $\bar{e}$ with the orientation so that e points to v, respectively c and we define $\mathfrak{U}_e := \mathfrak{U}_i \cap \mathfrak{U}_{i'}$. Finally, define

$$M_g := (\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}})^{\tau}_{|\Omega_g} \overset{\text{Cor. 9.20}}{\cong} \mathrm{Sym}^n(\mathrm{Hom}(\Lambda_g, \Omega_A))$$

for any $g \in \mathrm{GL}_2(\mathbb{A}^f)$. This clearly yields a local system $(M := M_g \otimes_A K, M_g)$.

Lemma 10.4 *Suppose $\mathfrak{X}$ is a connected rigid space and $j : \mathfrak{U} \to \mathfrak{X}$ a Zariski open immersion with $\mathfrak{U} \neq \mathfrak{X}$. If $\tilde{\underline{\mathcal{F}}}$ is a rigid τ-sheaf on $\mathfrak{X}$ over $\mathfrak{D}_A$ which is nilpotent on $\mathfrak{X} - \mathfrak{U}$. Then $\tilde{\underline{\mathcal{F}}}^{\tau} = 0$.*

In particular $(j_{\mathcal{K}!}(\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}})_{|\mathfrak{U}_c})^{\tau} = 0$ for each cusp c of $\overline{\mathfrak{M}}_{\mathcal{K}}$.

Proof: The second assertion clearly follows from the first. So let $\mathcal{I}$ be an ideal sheaf for a complement $i \colon \mathfrak{Z} \to \mathfrak{X}$ of $\mathfrak{X} \smallsetminus \mathfrak{U}$. Choose $m > 0$ such that $\tau^m = 0$ on $i^*\tilde{\underline{\mathcal{F}}}$. Suppose now that u is a τ-invariant of $\tilde{\underline{\mathcal{F}}}$. Then, upon repeatedly applying τ^m, we find that u lies in $\mathcal{I}^{q^n}\tilde{\underline{\mathcal{F}}}$ for all $n > 0$. But the intersection of these sheaves is clearly zero, and hence $u = 0$, as asserted. ∎

The affinoids $\mathfrak{U}_v$ and $\mathfrak{U}_{\bar{e}}$ can be described as suitable affinoids of the $\Omega_{x_\nu} \subset \Omega_{\mathcal{K}}$, namely those above the edge e, respectively vertex v of $\mathcal{T}_{x_\nu}$. To make the notation less confusing, we write $\mathfrak{U}'_t$ for the affinoid in $\Omega_{\mathcal{K}}$ corresponding to a simplex $t \in \mathcal{T}_{\mathcal{K}}$, and we use the isomorphisms $\mathfrak{U}_t \cong \mathfrak{U}'_t$ for $t \in R_0 \cup R_1$ to describe the non-cuspidal affinoids of the cover $\bar{\mathfrak{U}}_{\mathcal{K}}$. Furthermore, for $i_0 < i_1$, we identify the intersection $\mathfrak{U}_{i_0} \cap \mathfrak{U}_{i_1}$ with the corresponding affinoid subdomain of $\mathfrak{U}'_{i_1} \subset \Omega_{\mathcal{K}}$, which we call $\mathfrak{U}'_{i_0 i_1}$.

Thus given a j-cochain c_j, we can define a value for each translate $\gamma\mathfrak{U}'_{i_0\ldots i_j}$ as $\gamma c_j(\mathfrak{U}_{i_0} \cap \ldots \mathfrak{U}_{i_j})$. This can be used to explicitly describe the restriction maps needed for the Čech cover $\bar{\mathfrak{U}}_{\mathcal{K}}$ in terms of the combinatorics and restriction maps on $\Omega_{\mathcal{K}}$. Let us look at the restriction maps that occur in the first differential of the Čech complex. Assume that we have $v = i < \bar{e} = i'$ for some vertex v which is 'adjacent' to an edge e, i.e., so that we have $t(e) = \gamma_e v_e$ and $v_e = v$. Consider the diagram

$$\begin{tikzcd}
\mathfrak{U}_{\bar{e}} \arrow[d, "\cong"] & & \mathfrak{U}_i \cap \mathfrak{U}_{i'} = \mathfrak{U}_e \arrow[ll, hook'] \arrow[rr, hook] \arrow[dl, "\cong"] \arrow[dr, "\cong"] & & \mathfrak{U}_v \arrow[d, "\cong"] \\
\mathfrak{U}'_{\bar{e}} & \mathfrak{U}'_{\bar{e}} \cap \mathfrak{U}'_{t(e)} \arrow[l, hook'] \arrow[rr, "\cong", "\cdot\gamma_e"'] & & \mathfrak{U}'_{\gamma_e\bar{e}} \cap \mathfrak{U}'_v \arrow[r, hook] & \mathfrak{U}'_v .
\end{tikzcd}$$

On $\Omega_{\mathcal{K}}$ the restriction map from an affinoid to an affinoid subdomain for the module

$$(\mathrm{Sym}^n(\tilde{\underline{\mathcal{F}}}_{\Omega_{\mathcal{K}}})^{\mathfrak{D}_A\text{-rig}}_{\Omega_g})^{\tau} \cong \mathrm{Sym}^n(\mathrm{Hom}_A(\Lambda_g, \Omega_A))$$

is simply the identity. Furthermore the transition from $\mathfrak{U}'_{\bar{e}} \cap \mathfrak{U}'_{t(e)}$ to $\mathfrak{U}'_{\gamma_e\bar{e}} \cap \mathfrak{U}'_v$ is given by multiplication with γ_e. Thus the horizontal maps on the bottom are well understood. The vertical arrows and the down-left arrow are used to identify the sections on $\mathfrak{U}_{\bar{e}}$, $\mathfrak{U}_v$ and $\mathfrak{U}_e$. Based on this, we leave it to the reader to verify the following simple fact:

Say c_i are i-cochain for $i = 0, 1$ such that c_1 is the boundary of c_0. Then for $e \in R^o_{1,\nu}$, which say corresponds to $i < i' = \bar{e}$, we have

$$c_1(\mathfrak{U}_e) = \begin{cases} c_0(\mathfrak{U}_{\bar{e}}) & \text{if } i \text{ is a cusp,} \\ c_0(\mathfrak{U}_{\bar{e}}) - \gamma_e c_0(\mathfrak{U}_{v_e}) & \text{otherwise.} \end{cases} \tag{48}$$

Thus we obtained the following commutative diagram for the τ-invariants of the Čech cohomology for the sheaf $j_{\mathcal{K}!}(\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}})$ and the cover $\bar{\mathfrak{U}}_{\mathcal{K}}$:

$$\begin{tikzcd} \mathcal{C}^0(\bar{\mathfrak{U}}_{\mathcal{K}}, j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}})^\tau \arrow[r] \arrow[d, "\cong"] & \mathcal{C}^1(\bar{\mathfrak{U}}_{\mathcal{K}}, j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}})^\tau \arrow[d, "\cong"] \\ \bigoplus_\nu \Big(\bigoplus_{v\in R_0} M_{x_\nu} \oplus \bigoplus_{v \in R_1} M_{x_\nu} \arrow[r] & \bigoplus_{e \in R_1^o} M_{x_\nu} \Big), \end{tikzcd}$$

where the boundary map at the bottom is described by (48).

Comparing this with the description of $\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},\bullet}, \bar{M})$ and using Theorem 8.50 for the comparison of algebraic crystals and $\mathfrak{D}_A$-crystals, we have shown the following proposition:

Proposition 10.5 *There is an isomorphism of complexes*

$$\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},\bullet}, \bar{M}) \longrightarrow \mathcal{C}^\bullet(\bar{\mathfrak{U}}_{\mathcal{K}}, j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}})^\tau.$$

By invoking Lemma 5.47, the only non-vanishing cohomology of the complex on the left is $\Big(\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)\Big)^*$ in degree one. Because any representing τ-sheaf of $j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)})$ can be twisted arbitrarily often with the ideal sheaf of the cusps, one has $H^0(\bar{\mathfrak{U}}_{\mathcal{K}}, j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}) = 0$. Hence the right hand side in the above proposition has its only non-vanishing cohomology in degree one. As taking τ-invariants is left exact, this cohomology injects into $H^1(\bar{\mathfrak{U}}_{\mathcal{K}}, j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}})^\tau$. Theorem 8.50 implies that the latter is isomorphic to

$$\Big(H^1(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty}, j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}}\Big)^\tau,$$

and hence we have shown:

Corollary 10.6 *The above isomorphism of complexes induces an injective map of A-modules*

$$\Big(\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)\Big)^* \hookrightarrow \Big(\underline{\mathcal{S}}_n^{\mathfrak{D}_A\text{-rig}}(\mathcal{K})\Big)^\tau.$$

Definition 10.7 *The map in the previous corollary is called the* Eichler-Shimura map *for Drinfeld modular forms of level $\mathcal{K}$ and weight n.*

10.2 Torsion points

It remains to show that the Eichler-Shimura map is an isomorphism. The essential tool will be an analysis of the torsion points of $\underline{\varphi}_{\mathcal{K}}$ and a result of Pink, [40].

We first start with a lemma on $\underline{\mathcal{S}}_n(\mathcal{K})$.

Lemma 10.8 *The crystal $\underline{\mathcal{S}}_n(\mathcal{K})$ is representable by a τ-module (N, τ_N) on K_∞ over A such that N is finitely generated and projective over $K_\infty \otimes_k A$, say of rank $r \in \mathbb{N}_0$, and such that one has a short exact sequence*

$$0 \longrightarrow {K_\infty}^\sigma \otimes_{K_\infty} N \xrightarrow{\tau_N} N \longrightarrow C \longrightarrow 0, \tag{49}$$

for some τ-module (C, τ_C) such that C is finite over K_∞.

Proof: By its definition and Theorem 7.13 (d), the crystal $\underline{\mathcal{S}}_n(\mathcal{K})$ is of pullback type. Thus we can represent it by a τ-module (N', τ'), where $N' \cong K_\infty \otimes_k A^{r'}$. We consider the sequence of $K_\infty \otimes A$-linear maps

$$0 \longrightarrow C' \longrightarrow {K_\infty}^\sigma \otimes_{K_\infty} N' \xrightarrow{\tau'} N' \longrightarrow C'' \longrightarrow 0.$$

If τ' is not injective, then the image of τ' is of smaller rank as N'. Since $(\operatorname{Im}\tau', \tau'_{|\operatorname{Im}\tau'})$ is nil-isomorphic to (N', τ'), an inductive argument yields the desired (N, τ_N). ∎

Lemma 10.9 *Let $\mathfrak{n}$ be an ideal of A and $\mathcal{D}_A$ the coordinate ring of $\mathfrak{D}_A$. Then $\mathcal{D}_A/\mathfrak{n}\mathcal{D}_A \cong K_\infty \otimes_A A/\mathfrak{n}$.*

The proof is easily reduced to the case $A = k[T]$. In this situation, the important observation is that points of $\operatorname{Spec} k[t] \smallsetminus \{0\}$ lie on the rim of the unit disc $\operatorname{Spm} K_\infty\langle\langle T\rangle\rangle$. Details are left to the reader.

The following simple lemma bounds the rank of the τ-invariants in terms of the rank of the underlying module.

Lemma 10.10 *Suppose (N, τ) is a locally free τ-module on K_∞ over A of rank r. Then for each proper non-zero ideal $\mathfrak{n}$ of A, the $A/\mathfrak{n}$-module $(N \otimes_{K_\infty \otimes A} (K_\infty^{\mathrm{alg}} \otimes A/\mathfrak{n}))^\tau$ is free of some rank $r_\mathfrak{n} \le r$, and $((N, \tau)^{\mathfrak{D}_A\text{-rig}})^\tau$ is a projective A-module of rank at most $\min_\mathfrak{n} r_\mathfrak{n}$.*

Proof: Fix an ideal $\mathfrak{n}$ as above. By Lang's theorem, cf. [1], proof of Lem. 1.8.2, it easily follows that $(N \otimes_{K_\infty \otimes A} (K_\infty^{\mathrm{alg}} \otimes A/\mathfrak{n}))^\tau$ is free over $A/\mathfrak{n}$ of some rank $r_\mathfrak{n} \le r$. By the above lemma, $\mathcal{D}_A/\mathfrak{n}\mathcal{D}_A \cong K_\infty \otimes A/\mathfrak{n}$, and furthermore

$$(N, \tau)^{\mathfrak{D}_A\text{-rig}} \otimes \mathcal{D}_A/\mathfrak{n}\mathcal{D}_A \cong (N, \tau) \otimes_A A/\mathfrak{n}.$$

Since $N^{\mathfrak{D}_A\text{-rig}}$ is A-torsion free, the set of its τ-invariants is A-torsion free. Assume that its rank was greater than r. Then the set of τ-invariants of $N \otimes_A A/\mathfrak{n}$ would contain an $A/\mathfrak{n}$-module of rank at least $r + 1$. This contradicts the bound $r_\mathfrak{n} \le r$. ∎

Proposition 10.11 *Let $\underline{N}$ be the τ-module (N, τ_N) of Lemma 10.8. Then for each $\mathfrak{p} \in \operatorname{Max}(A)$, the induced morphism τ on $\underline{N} \otimes_A A/\mathfrak{p}$ is an isomorphism and furthermore $\underline{N}^{\mathfrak{D}_A\text{-rig}}$ is uniformizable.*

Proof: Let r denote the rank of N. By Lemma 10.10 we know that $(N^{\mathfrak{D}_A\text{-rig}})^\tau$ is a projective A-module of rank $r' \le r$. It contains the submodule $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)$ whose rank we denote by r''. We claim that $r'' = r$.

To prove the claim, we fix any proper non-zero ideal $\mathfrak{n}$ of A and consider the étale sheaf $\mathsf{F} := \mathfrak{b}_{K_\infty}^*((j_!\underline{\mathcal{F}}_\mathcal{K}^{(n-2)}) \otimes A/\mathfrak{n})_{\text{ét}}$. By Corollary 7.20, this is simply the sheaf of $A/\mathfrak{n}$-torsion points of $\underline{\varphi_\mathcal{K}}$, raised to the $(n-2)$-th symmetric power and extended by zero at the cusps. Because K_∞ is of generic characteristic, the sheaf of $\mathfrak{n}$-torsion points corresponds to a Galois cover of $\overline{\mathfrak{M}}_{\mathcal{K}, K_\infty}$.

By the previous lemma, $(\underline{\mathcal{S}}_n(\mathcal{K}) \otimes A/\mathfrak{n})_{\text{ét}}$ has global sections on K_∞^{alg} with rank at most r. By Theorem 7.18, which described the functorialities of $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}$, we have

$$(\underline{\mathcal{S}}_n(\mathcal{K}) \otimes A/\mathfrak{n})_{\text{ét}}(\operatorname{Spec} K_\infty^{\mathrm{alg}}) \cong H^1_{\text{ét}}(\overline{\mathfrak{M}}_{\mathcal{K}, K_\infty^{\mathrm{alg}}}, \mathsf{F}).$$

We claim that $H^1_{\text{ét}}(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty},\mathsf{F})$ is free over $A/\mathfrak{n}$ of rank r'', independently of $\mathfrak{n}$. Let us see how this second claim implies the first one. The sequence (49) shows that for almost all maximal ideals $\mathfrak{p}$, the map τ on $N\otimes_A A/\mathfrak{p}$ is an isomorphism. Hence for $\mathfrak{n}=\mathfrak{p}$ it follows that $r=r''$, as asserted and we have shown uniformizability. But the second claim together with $r=r''$ also implies our second assertion, and so to complete the proof of the proposition, it remains to prove the second claim.

For this, we quote the following result from Pink, [40], Thm. 0.2, Def. 5.3, Prop. 5.6 (a).

Theorem 10.12 *Suppose X is an irreducible smooth projective curve over an algebraically closed field of characteristic p. Let G be a constructible étale sheaf of k-vector spaces on X. Let $\chi(X,\mathsf{G})$ denote the Euler-Poincaré characteristic of G. Assume that there exists an irreducible finite Galois cover $\pi\colon Y\to X$ such that*

(a) the generic monodromy of $\pi^*\mathsf{G}$ is a p-group,

(b) Y is ordinary,

(c) all local ramification groups at $x\in X$ of the generic fiber G_η are p-groups.

Assume further that there is a dense open immersion $j\colon U\to X$, such that $j^*\mathsf{G}$ is lisse and $\mathsf{G}\cong j_!j^*\mathsf{G}$. Let $h(X)=\operatorname{card}(X\smallsetminus U)$. Then

$$\chi(X,\mathsf{G})=(1-g(X)-h(X))\dim\mathsf{G}_\eta.$$

As explained in [40], p. 3, the schemes $X:=\overline{\mathfrak{M}}_{\mathcal{K},K_\infty^{\text{alg}}}$ are ordinary for any admissible $\mathcal{K}$. Furthermore if $\mathcal{K}'\lhd\mathcal{K}$ are admissible, then the induced morphism

$$\pi_{\mathcal{K},\mathcal{K}'}\colon\overline{\mathfrak{M}}_{\mathcal{K}',K_\infty^{\text{alg}}}\longrightarrow\overline{\mathfrak{M}}_{\mathcal{K},K_\infty^{\text{alg}}}$$

is a Galois cover with Galois group $\mathcal{K}/\mathcal{K}'$. In particular, the latter is satisfied for the normal subgroup $\mathcal{K}(\mathfrak{n})\cap\mathcal{K}$ of $\mathcal{K}$ for any level $\mathfrak{n}$-structure. Moreover the ramification of $\pi_{\mathcal{K},\mathcal{K}'}$ occurs only at the cusps and is a finite p-group. Furthermore, for F as above, the generic monodromy of $\pi^*_{\mathcal{K},\mathcal{K}(\mathfrak{n})\cap\mathcal{K}}\mathsf{F}$ is trivial. Hence we find that

$$\chi(X,\mathsf{F})=(1-g(X)-h(X))\dim\mathsf{F}_\eta,$$

with $X=\overline{\mathfrak{M}}_{\mathcal{K},K_\infty^{\text{alg}}}$, and where $h(X)$ is the number of cusps of X and $\dim\mathsf{F}_\eta=(n-1)\dim_k A/\mathfrak{n}$. Because $H^0_{\text{ét}}(X,\mathsf{F})=0$, we have shown that

$$\dim_k H^1_{\text{ét}}(X,\mathsf{F})=(n-1)(g(X)+h(X)-1)\dim_k A/\mathfrak{n}.$$

By Proposition 5.4 and the comparison isomorphisms in Section 5, we also have

$$\operatorname{rank}_A\mathbf{C}^{\text{St}}(\mathcal{K},A)=(n-1)(g(X)+h(X)-1)$$

for the projective A-module $\mathbf{C}^{\text{St}}(\mathcal{K},A)$. Therefore the k-dimension of the module $\mathbf{C}^{\text{St}}(\mathcal{K},A)\otimes_A A/\mathfrak{n}$ is given by $(n-1)(g(X)+h(X)-1)\dim_k A/\mathfrak{n}$. As in the proof of Lemma 10.10, the module $\mathbf{C}^{\text{St}}(\mathcal{K},A)\otimes_A A/\mathfrak{n}$ is a submodule of $H^1_{\text{ét}}(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty},\mathsf{F})$, which in turn is a submodule of $H^1_{\text{ét}}(X,\mathsf{F})$. Hence by counting dimensions, all these modules must agree, and in particular the second claim is shown. ∎

As a corollary to the above proposition, we obtain the following:

Corollary 10.13 *The crystals $\mathfrak{b}^*_{K_\infty}\underline{\mathcal{S}}_n(\mathcal{K})$ on $\operatorname{Spec} K_\infty$ are uniformizable in the sense of Definition 9.17.*

For the proof of Theorem 10.3, we need the following lemma:

Lemma 10.14 *For $\tilde{\underline{\mathcal{F}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, \mathfrak{D}_A)$ we consider the morphism*

$$\tilde{\underline{\mathcal{F}}}^\tau \otimes_A (A/\mathfrak{n}A) \xrightarrow{\alpha} (\tilde{\underline{\mathcal{F}}} \otimes_{\mathfrak{D}_A} \mathfrak{D}_A/\mathfrak{n}\mathfrak{D}_A)^\tau,$$

where $\mathfrak{n} \neq 0$ is an ideal of A. The following assertions hold:

(i) If $\tilde{\underline{\mathcal{F}}}$ is flat over A. Then α is a monomorphism.

(ii) If $\tilde{\underline{\mathcal{F}}} = \tilde{\mathbb{1}}_{\mathfrak{X},\mathfrak{D}_A}$, then α is an isomorphism.

(iii) If $\mathfrak{X}$ is connected and there is a point $x \in \mathfrak{X}$ such that the fiber of $\tilde{\underline{\mathcal{F}}}$ at x is nilpotent, then the domain and range of α are both zero.

Proof: For part (i), we may assume that $\mathfrak{n}$ is principal, say generated by (a). Let us first show that

$$\tilde{\underline{\mathcal{F}}}^\tau \cap \Gamma(\mathfrak{X}, a\tilde{\underline{\mathcal{F}}}) = a(\tilde{\underline{\mathcal{F}}})^\tau.$$

The inclusion $\supset$ is obvious, and to show $\subset$, we let f be an element of the left hand side. This means that there exists $g \in \Gamma(\mathfrak{X}, \tilde{\underline{\mathcal{F}}})$ such that $ag = f$ and that $f^\tau = f$. But then $a(g - g^\tau) = 0$ and hence $g = g^\tau$ because $\tilde{\underline{\mathcal{F}}}$ is A-torsion free. This shows that $f = ag$ lies in the right hand side. By what we have just shown, it follows that $(a\tilde{\underline{\mathcal{F}}})^\tau = a(\tilde{\underline{\mathcal{F}}})^\tau$. The assertion in (i) now follows by applying the Snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & a\tilde{\underline{\mathcal{F}}} \arrow[r] \arrow[d, "1-\tau"] & \tilde{\underline{\mathcal{F}}} \arrow[r] \arrow[d, "1-\tau"] & \tilde{\underline{\mathcal{F}}} \otimes_{\mathfrak{D}_A} \mathfrak{D}_A/a\mathfrak{D}_A \arrow[r] \arrow[d, "1-\tau"] & 0 \\
0 \arrow[r] & a\tilde{\underline{\mathcal{F}}} \arrow[r] & \tilde{\underline{\mathcal{F}}} \arrow[r] & \tilde{\underline{\mathcal{F}}} \otimes_{\mathfrak{D}_A} \mathfrak{D}_A/a\mathfrak{D}_A \arrow[r] & 0.
\end{tikzcd}$$

Part (ii) is a simple explicit computation and left to the reader. In part (iii), the domain of α is zero by Lemma 10.4. To show that the range is zero as well, one needs to prove an obvious modification of Lemma 10.4. Details are left to the reader. ∎

Proof of Theorem 10.3: Let $\mathfrak{n}$ be a non-zero ideal of A. We use the following abbreviations: $\tilde{\underline{\mathcal{M}}}^i := \mathcal{C}^i(\bar{\mathfrak{U}}_\mathcal{K}, j_{\mathcal{K}!}\underline{\mathcal{F}}_\mathcal{K}^{(n),\mathfrak{D}_A\text{-rig}})$, $\tilde{\underline{\mathcal{M}}}^i_{/\mathfrak{n}} := \tilde{\underline{\mathcal{M}}}^i \otimes_{\mathfrak{D}_A} \mathfrak{D}_A/\mathfrak{n}\mathfrak{D}_A$ and $\underline{\mathcal{H}} := H^1(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty}, \mathfrak{b}^*_{K_\infty} j_{\mathcal{K}!}\underline{\mathcal{F}}_\mathcal{K}^{(n)})$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & (\underline{\mathcal{M}}^0)^\tau \otimes_A A/\mathfrak{n} \arrow[r] \arrow[d, "\cong"] & (\underline{\mathcal{M}}^1)^\tau \otimes_A A/\mathfrak{n} \arrow[r] \arrow[d, "\cong"] & \left(\underline{\mathcal{H}}^{\mathfrak{D}_A\text{-rig}}\right)^\tau \otimes_A A/\mathfrak{n} \arrow[d, hook] \\
0 \arrow[r] & (\underline{\mathcal{M}}^0_{/\mathfrak{n}})^\tau \arrow[r] & (\underline{\mathcal{M}}^1_{/\mathfrak{n}})^\tau \arrow[r] & \left(\underline{\mathcal{H}} \otimes_{\mathfrak{D}_A} \mathfrak{D}_A/\mathfrak{n}\mathfrak{D}_A\right)^\tau,
\end{tikzcd}$$

where by the previous lemma, the left and middle vertical maps are isomorphisms and the right vertical map is a monomorphism.

In the previous proof, we had shown that $(\underline{\mathcal{H}} \otimes_{\mathfrak{D}_A} \mathfrak{D}_A/\mathfrak{n}\mathfrak{D}_A)^\tau$ is a free $A/\mathfrak{n}$-module of the same rank as $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)^*$. Because the latter module is flat over A, the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},0}, \bar{M}) \longrightarrow \mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},1}, \bar{M}) \longrightarrow \mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)^* \longrightarrow 0.$$

remains exact after tensoring with $A/\mathfrak{n}$ over A. Let us denote this latter sequence temporarily by $(*)$. Then the left and middle terms in $(*)$ are isomorphic to the left and middle terms of the left exact sequences displayed in the diagram above. Therefore counting dimensions over k, it follows that the second row in the above diagram is short exact. But then clearly the first row must be short exact as well, and this for all non-zero ideals $\mathfrak{n}$ of A. This implies that

$$0 \longrightarrow (M^0)^\tau \longrightarrow (M^1)^\tau \longrightarrow (H^1(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty}, j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\mathfrak{D}_A\text{-rig}})^\tau \longrightarrow 0$$

is exact, and hence the monomorphism in Corollary 10.6 is an isomorphism. ∎

Remark 10.15 Recall that $s_n(\mathcal{K})$ is the dimension of $\underline{S}_n(\mathcal{K})$. The above proof shows $((b^*_{K_\infty}\underline{\mathcal{S}}_n(\mathcal{K})) \otimes A/\mathfrak{n})_{\text{ét}}$ is locally free over $A/\mathfrak{n}$ of rank $s_n(\mathcal{K})$, and furthermore that any locally free τ-sheaf which represents $(b^*_K\underline{\mathcal{S}}_n(\mathcal{K}))$ and on which τ is injective, is of rank $s_n(\mathcal{K})$, too.

11 Maximal extensions

In this section we will develop a notion necessary to formulate, in the following section, an Eichler-Shimura isomorphism for double cusp forms, namely the notion of *maximal extension* of a coherent τ-sheaf, respectively crystal.

Let $j: U \to X$ be an open immersion. The idea is that a crystal $\underline{\mathcal{F}}$ on U has two natural extensions to X. One extension is $j_!\underline{\mathcal{F}}$, which is in an obvious sense the smallest one. The other is the direct limit $j_{\#}\underline{\mathcal{F}}$ of all coherent τ-subsheaves of $j_*\underline{\mathcal{F}}$, where we represent $\underline{\mathcal{F}}$ by a τ-sheaf. The latter is called the maximal extension provided it is coherent. The functors $j_!$ and $j_{\#}$ have properties similar to the corresponding functors $j_!$ and j_* in étale cohomology.

Building on work of Gardeyn, [13], Ch. 1, in Subsection 11.1 we introduce the main concepts and prove some basic results. In particular, we give a criterion for $j_{\#}\underline{\mathcal{F}}$ to be coherent. In Subsection 11.2, we will study the maximal extension of the universal τ-sheaf $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ under the open immersion $j_{\mathcal{K}}\colon \mathfrak{M}_{\mathcal{K}} \hookrightarrow \overline{\mathfrak{M}}_{\mathcal{K}}$. This yields the τ-sheaf $j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ which, considered as a crystal, contains $j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$. The discrepancy will turn out to be the essentially the unit crystal supported on the cusps. Again, some of the basic results we need are due to Gardeyn, [13], Ch. 1 and 6.

Subsection 11.3 compares $j_{\#}$ for the étale and crystalline sites. In Subsection 11.4 we briefly discuss maximal extensions for rigid analytic τ-sheaves. This is applied in the last subsection to compute the τ-invariants of the maximal extension of $\mathrm{Sym}^n\,\underline{\tilde{\mathcal{F}}}(\mathcal{K})$ near the cusps.

11.1 Maximal extensions of τ-sheaves and crystals

Let us fix throughout this subsection an open immersion $j: U \to X$, a closed complement $i: Z \to \bar{U}$ in the Zariski closure $\bar{U} \subset X$ of U and a coherent τ-sheaf (respectively crystal) $\underline{\mathcal{F}}$ on U. Motivated by some work of Gardeyn, [13], Ch. 1, we will introduce the notion of a maximal extension of $\underline{\mathcal{F}}$. Moreover we provide some results on the existence of maximal extensions that will be needed in the subsequent sections.

The definition of a maximal extension, given below, generalizes [13], Def. 1.12, and is modeled after the Néron mapping property.

Definition 11.1 *Any $\underline{\mathcal{G}} \in \mathbf{Coh}_\tau(X, A)$ with $j^*\underline{\mathcal{G}} \cong \underline{\mathcal{F}}$ is called an* extension *of $\underline{\mathcal{F}}$.*

We define $j_{\#}\underline{\mathcal{F}} \subset j_*\underline{\mathcal{F}} \in \mathbf{QCoh}_\tau(X, A)$ as the union of all extensions $\underline{\mathcal{G}}$ of $\underline{\mathcal{F}}$, or equivalently as the union of all coherent τ-subsheaves of $j_*\underline{\mathcal{G}}$.

If $j_{\#}\underline{\mathcal{F}}$ is coherent, it is called the maximal extension of $\underline{\mathcal{F}}$ with respect to j.

If a maximal extension exists, it is unique up to unique isomorphism.

Remark 11.2 For $\underline{\mathcal{F}} = (\mathcal{F}, 0)$, one has $j_{\#}\underline{\mathcal{F}} = (j_*\mathcal{F}, 0)$ and thus in general $j_{\#}\underline{\mathcal{F}}$ is not coherent. Note also that for $\mathrm{codim}_X Z = 1$, the functor $\underline{\mathcal{F}} \mapsto j_{\#}\underline{\mathcal{F}}$ is not exact.

The following gives an intrinsic characterization of $j_{\#}\underline{\mathcal{F}}$:

Proposition 11.3 *A coherent τ-sheaf $\underline{\mathcal{G}}$ on X is a maximal extension of $\underline{\mathcal{F}}$ if and only if for all $\underline{\mathcal{H}} \in \mathbf{Coh}_\tau(X, A)$, the canonical map*

$$\mathrm{Hom}_{\mathbf{Coh}_\tau(X,A)}(\underline{\mathcal{H}}, \underline{\mathcal{G}}) \longrightarrow \mathrm{Hom}_{\mathbf{Coh}_\tau(U,A)}(j^*\underline{\mathcal{H}}, \underline{\mathcal{F}})$$

is an isomorphism.

Proof: It clearly suffices to show that $j_\#\underline{\mathcal{F}}$ is characterized by the properties that a) for all $\underline{\mathcal{H}} \in \mathbf{Coh}_\tau(X,A)$ the canonical map

$$\mathrm{Hom}_{\mathbf{QCoh}_\tau(X,A)}(\underline{\mathcal{H}}, j_\#\underline{\mathcal{F}}) \longrightarrow \mathrm{Hom}_{\mathbf{Coh}_\tau(U,A)}(j^*\underline{\mathcal{H}}, \underline{\mathcal{F}})$$

is an isomorphism and b) $j_\#\underline{\mathcal{F}}$ is an inductive limit of coherent τ-sheaves.

By its very definition $j_\#\underline{\mathcal{F}}$ satisfies b). For a), observe that the isomorphism holds for $j_*\underline{\mathcal{F}}$ in place of $j_\#\underline{\mathcal{F}}$. Because all the $\underline{\mathcal{H}}$ are coherent, and hence so is their image in $j_*\underline{\mathcal{F}}$, the isomorphism holds for $j_\#\underline{\mathcal{F}}$ as well. It remains to show that $j_\#\underline{\mathcal{F}}$ is determined by a) and b) up to unique isomorphism.

Let $\underline{\mathcal{G}}'$ be another quasi-coherent τ-sheaf which satisfies a) and b). Without loss of generality we assume $\underline{\mathcal{G}}' = \varinjlim \underline{\mathcal{G}}_i$ where the $\underline{\mathcal{G}}_i$ are coherent subsheaves of $\underline{\mathcal{G}}'$. By a) for $\underline{\mathcal{G}}'$ and for $j_\#\underline{\mathcal{F}}$, we have for any $\underline{\mathcal{G}}_i$ an isomorphism

$$\mathrm{Hom}_{\mathbf{QCoh}_\tau(X,A)}(\underline{\mathcal{G}}_i, j_\#\underline{\mathcal{F}}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Coh}_\tau(X,A)}(\underline{\mathcal{G}}_i, \underline{\mathcal{G}}'). \tag{50}$$

Let $\varphi_i : \underline{\mathcal{G}}_i \to j_\#\underline{\mathcal{F}}$ be the homomorphism that corresponds to the canonical inclusion $i_i : \underline{\mathcal{G}}_i \to \underline{\mathcal{G}}'$. It is easy to see that the φ_i form an inductive system, and hence there is an induced homomorphism $\varphi : \underline{\mathcal{G}}' \to j_\#\underline{\mathcal{F}}$. Let $\underline{\mathcal{G}}_i'$ be the kernel of φ_i and $i_i' : \underline{\mathcal{G}}_i' \to \underline{\mathcal{G}}$ the corresponding monomorphism. Then it follows from the isomorphism in (50) for $\underline{\mathcal{G}}_i'$ instead of $\underline{\mathcal{G}}_i$ that i_i' must be zero. This shows that φ is a monomorphism.

Reversing the roles of $\underline{\mathcal{G}}'$ and $j_\#\underline{\mathcal{F}}$, one obtains conversely a monomorphism $\varphi' : \underline{\mathcal{G}}' \hookrightarrow j_\#\underline{\mathcal{F}}$. By considering coherent subobjects of $\underline{\mathcal{G}}'$ and $j_\#\underline{\mathcal{F}}$, it is not hard to see, that the composites $\varphi'\varphi$ and $\varphi\varphi'$ are the identity. This proves the assertion. ∎

Definition 11.4 *A crystal $\underline{\mathcal{G}} \in \mathbf{Crys}(X,A)$ is called an extension of $\underline{\mathcal{F}}$ if $j^*\underline{\mathcal{G}} \cong \underline{\mathcal{F}}$. It is called a* maximal extension *if in addition for all $\underline{\mathcal{H}} \in \mathbf{Crys}(X,A)$, the canonical map*

$$\mathrm{Hom}_{\mathbf{Crys}(X,A)}(\underline{\mathcal{H}}, \underline{\mathcal{G}}) \longrightarrow \mathrm{Hom}_{\mathbf{Crys}(U,A)}(j^*\underline{\mathcal{H}}, \underline{\mathcal{F}})$$

is an isomorphism.

Note that in the category of crystals there always exists a unique minimal extension of $\underline{\mathcal{F}}$, namely $j_!\underline{\mathcal{F}}$.

Proposition 11.5 *Assume that $\underline{\mathcal{G}} \in \mathbf{Coh}_\tau(X,A)$ is a maximal extension of $\underline{\mathcal{F}}$. Then the crystal represented by $\underline{\mathcal{G}}$ is a maximal extension of the crystal represented by $\underline{\mathcal{F}}$.*

Proof: Given $\varphi : j^*\underline{\mathcal{H}} \dashrightarrow \underline{\mathcal{F}}$, we need to construct an extension $\tilde{\varphi} : \underline{\mathcal{H}} \dashrightarrow \underline{\mathcal{G}}$ and show that it is unique in the category of crystals. So let φ be given. Then for some $n \in \mathbb{N}$, there exists a diagram of τ-sheaves

$$j^*\underline{\mathcal{H}} \xLeftarrow{\tau^n} j^*((\sigma_X^n \times \mathrm{id})^*\underline{\mathcal{H}}) \cong (\sigma_U^n \times \mathrm{id})^*(j^*\underline{\mathcal{H}}) \xrightarrow{\varphi_n} \underline{\mathcal{F}}$$

which represents φ. By the maximality of $\underline{\mathcal{G}}$ as a τ-sheaf there exists a unique map of τ-sheaves $\tilde{\varphi}_n : (\sigma_X^n \times \mathrm{id})^*\underline{\mathcal{H}} \to \underline{\mathcal{G}}$ which extends φ_n. Hence the induced diagram

$$\underline{\mathcal{H}} \xLeftarrow{\tau^n} (\sigma_X^n \times \mathrm{id})^*\underline{\mathcal{H}} \xrightarrow{\tilde{\varphi}_n} \underline{\mathcal{G}}$$

defines an extension of φ.

Because $\mathrm{Hom}(\underline{\mathcal{H}}, \underline{\mathcal{G}})$ is a group under addition, for the uniqueness it suffices to show that any extension $\tilde{\varphi}$ of $\varphi = 0$ is the zero morphism. Let $\tilde{\varphi}$ be such an extension. Then for some $m \in \mathbb{N}$ there exists a diagram in $\mathbf{Coh}_\tau(X, A)$

$$\underline{\mathcal{H}} \xLeftarrow{\tau^m} (\sigma_X^m \times \mathrm{id})^* \underline{\mathcal{H}} \xrightarrow{\tilde{\varphi}_m} \underline{\mathcal{G}}$$

which represents $\tilde{\varphi}$. Furthermore, if we define for all $n \geq m$ the map $\tilde{\varphi}_n := \tilde{\varphi}_m \circ (\sigma^{n-m} \times \mathrm{id})^* \tau^{n-m}$, then the diagram

$$\underline{\mathcal{H}} \xLeftarrow{\tau^n} (\sigma_X^n \times \mathrm{id})^* \underline{\mathcal{H}} \xrightarrow{\tilde{\varphi}_n} \underline{\mathcal{G}}$$

also represents $\tilde{\varphi}$. Defining $\varphi_n := j^* \tilde{\varphi}_n \colon (\sigma_U^n \times \mathrm{id})^* j^* \underline{\mathcal{H}} \to \underline{\mathcal{F}}$, the diagram

$$j^* \underline{\mathcal{H}} \xLeftarrow{\tau^n} (\sigma_U^n \times \mathrm{id})^* j^* \underline{\mathcal{H}} \xrightarrow{\varphi_n} \underline{\mathcal{F}}$$

clearly represents φ. Thus for $n \gg 0$ the map φ_n must be zero. Because $\underline{\mathcal{G}}$ is maximal, the extension $\tilde{\varphi}_n$ is unique and hence it must be zero as well, and the assertion follows. ∎

Remark 11.6 We do not know whether any crystal posseses a maximal extension. For instance the example $\underline{\mathcal{F}} = (\mathcal{F}, 0)$ of Remark 11.2, considered as a crystal, has zero as its maximal extension.

Next we give some examples of maximal extensions:

Proposition 11.7 *Suppose X is normal, $\mathrm{codim}_X Z \geq 2$ and $\mathcal{F}$ is locally free. Then $j_* \mathcal{F}$ is coherent, so that $j_\# \underline{\mathcal{F}} = j_* \underline{\mathcal{F}}$ is the maximal extension of $\underline{\mathcal{F}}$ to X.*

The coherence of $j_* \mathcal{F}$ is a well-known result. Alternatively, it follows from Lemma 11.13 which we prove below.

For some further examples, we introduce the following notion from [13], §1.2:

Definition 11.8 *A τ-sheaf $\underline{\mathcal{G}}$ on X is said to have* good reduction on Z *if the map*

$$\tau_{i^* \mathcal{G}} \colon (\sigma \times \mathrm{id})^* i^* \mathcal{G} \to i^* \mathcal{G}$$

is injective.

A locally free extension $\underline{\mathcal{G}}$ of $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U, A)$ is called good *if $\tau_{\mathcal{F}}$ is injective and $\underline{\mathcal{G}}$ has good reduction on Z. (In particular, good extensions can only exist if $\underline{\mathcal{F}}$ is locally free.)*

Proposition 11.9 *If X is normal, then any good extension is maximal.*

The proof will be given after the proof of Theorem 11.15. As a consequence, if $\underline{\mathcal{G}}$ is a τ-sheaf attached to a family of A-motives on X and U is a dense open subset of X, then $\underline{\mathcal{G}}$ is the maximal extension of the restriction $j^* \underline{\mathcal{G}}$. For later use, we state the following immediate corollary.

Corollary 11.10 *Suppose X is normal and $\bar{U} = X$. Then $j_\# \mathbb{1}_{U,A} = \mathbb{1}_{X,A}$. In particular, if $j^* \underline{\mathcal{F}} \supset \mathbb{1}_{U,A}$, then with respect to this inclusion, one has*

$$\underline{\mathcal{F}} \cap j_* \mathbb{1}_{U,A} \subset \mathbb{1}_{X,A}.$$

For the remainder of this subsection, we will give sufficient conditions for the existence of maximal extensions of coherent τ-sheaves. The proofs of the following two results are elementary and left to the reader.

Lemma 11.11 *Suppose $j' \colon V \to X$ is an open immersion. Consider the pull-back diagram*

$$\begin{tikzcd} U \arrow[r, "j"] & X \\ U \cap V \arrow[u, "\tilde{j}'"] \arrow[r, "\tilde{j}"'] & V . \arrow[u, "j'"'] \end{tikzcd}$$

Then $(j')^* j_\# \underline{\mathcal{F}} \cong \tilde{j}_\# (\tilde{j}')^* \underline{\mathcal{F}}$. Thus if $\underline{\mathcal{F}}$ has a maximal extension, then so does $(\tilde{j}')^* \underline{\mathcal{F}}$ and it is given by $(j')^* j_\# \underline{\mathcal{F}}$.

Proposition 11.12 *Let $\{U_i\}$ be an open cover of X. Suppose $\underline{\mathcal{G}}$ is an extension of $\underline{\mathcal{F}}$ such that each $\underline{\mathcal{G}}_{|U_i}$ is a maximal extension of $\underline{\mathcal{F}}_{|U_i \cap U}$. Then $\underline{\mathcal{G}}$ is a maximal extension of $\underline{\mathcal{F}}$.*

The following lemma is useful to prove that j_* preserves coherence in certain important cases.

Lemma 11.13 *Let W be a normal scheme. Let $j_i \colon V_i \hookrightarrow W$ be open subschemes such that $V := \cup V_i$ is dense in W. Let $x_1, \dots, x_r$ be those generic points of $W \smallsetminus V$ which are of codimension 1. Let $\mathcal{F}$ be a quasi-coherent torsion free sheaf on W such that all the sheaves $\mathcal{F}_{|V_i}$ and all the sheaves $\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_{W,x_j}$ are coherent. Then $\mathcal{F}$ is coherent.*

Proof: As coherence is a local property, we may assume that $W = \operatorname{Spec} R$ is affine, where R is some normal domain, and that W contains at most one of the x_i, which we denote by $\mathfrak{p} \in \operatorname{Spec} R$. Refining the V_i, we may assume that they are all of the form $\operatorname{Spec} R_{f_i}$ for suitable elements f_i of R. Let F denote the fraction field of R, and M the torsion free module corresponding to $\mathcal{F}$ — to be torsion free simply means that $M \to M \otimes_R F := M_F$ is a monomorphism.

Our assumptions imply that M_F is a finite F vector space, say of dimension n. We choose a basis $e_1, \dots, e_n$ of M_F over F. Choose $f_i, f_\mathfrak{p} \in R$ such that $f_i e_l \in M_i := M_{f_i}$ for all i and $f_\mathfrak{p} e_l \in M_\mathfrak{p}$ for all l. Let $f := f_\mathfrak{p} \prod f_i$. Then $f R^n_{f_i} \subset M_i$ for all i and $f R^n_\mathfrak{p} \subset M_\mathfrak{p}$. Because $(M_i)_F = M_F = (M_\mathfrak{p})_F$, one can also find $g \in R$ such that $(f/g) R^n_{f_i} \supset M_i$ and $(f/g) R^n_\mathfrak{p} \supset M_\mathfrak{p}$. But then

$$M \subset M_\mathfrak{p} \cap \bigcap M_i \subset (f/g)(R^n_\mathfrak{p} \cap \bigcap R^n_{f_i}) \subset (f/g) \bigcap_\mathfrak{q} R^n_\mathfrak{q} = (f/g) R^n$$

where $\bigcap_\mathfrak{q} R_\mathfrak{q}$ is indexed by all primes $\mathfrak{q}$ of height one and the right most equality uses that for a normal domain R one has $\bigcap_\mathfrak{q} R_\mathfrak{q} = R$, cf. [45], Thm. 38. ∎

Corollary 11.14 *Let $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U, A)$ be torsion free and assume that X is normal. Let $z_1, \dots, z_r$ denote those generic points of Z which are of height 1 in X. For each $l \in \{1, \dots, r\}$ consider the pullback diagram*

$$\begin{tikzcd} U \arrow[r, "j"] & X \\ \operatorname{Spec} \mathcal{O}_{X,z_l} \smallsetminus \{z_l\} \arrow[u, "f'_l"] \arrow[r, "j_l"'] & \operatorname{Spec} \mathcal{O}_{X,z_l} . \arrow[u, "f_l"'] \end{tikzcd}$$

Let $\underline{\mathcal{F}_l}$ denote the pullback of $\underline{\mathcal{F}}$ to $\operatorname{Spec} \mathcal{O}_{X,l} \smallsetminus \{z_l\}$. If each of the $\underline{\mathcal{F}_l}$ has a maximal extension to $\operatorname{Spec} \mathcal{O}_{X,z_l}$, then $\underline{\mathcal{F}}$ has a maximal extension to X.

PROOF: We have to show that $j_\#\underline{\mathcal{F}}$ is coherent. Clearly $j_\#\underline{\mathcal{F}} \subset j_*\underline{\mathcal{F}}$ is zero outside $\bar{U}$, and so we may assume that $X = \bar{U}$. Denote by $j_\#\mathcal{F}$ the sheaf underlying $j_\#\underline{\mathcal{F}}$. By Lemma 11.11 and a limit argument, we have that $f_l^* j_\#\underline{\mathcal{F}} \cong j_{l\#}\underline{\mathcal{F}}_l$. Since $j_{l\#}\underline{\mathcal{F}}_l$ is coherent, the sheaves $f_l^* j_\#\mathcal{F}$ are coherent. For a normal k-scheme X, the scheme $X \times \operatorname{Spec} A$ is normal as well, because $k \to A$ is smooth, cf. [45], Corollary 21.E. The previous lemma now yields that $j_\#\mathcal{F}$ is coherent, and thus the proof is complete. ∎

Building on results of Gardeyn, we can now prove the following:

Theorem 11.15 *Suppose X is a normal scheme and $\underline{\mathcal{F}}$ a torsion free τ-sheaf on U such that the morphism $(\sigma_U \times \mathrm{id})^*\mathcal{F} \xrightarrow{\tau_\mathcal{F}} \mathcal{F}$ is injective. Then $j_\#\underline{\mathcal{F}}$ is coherent, i.e., $\underline{\mathcal{F}}$ has a maximal extension to X. Furthermore, if X is a curve and $\underline{\mathcal{F}}$ is locally free, then $j_\#\underline{\mathcal{F}}$ is locally free of the same rank.*

For the convenience of the reader, and because we make repeatedly use of them, we recall some results from [34] and [13]. The following lemma is a simple adaption of [34], Prop. 6.

Lemma 11.16 *Let R be a noetherian integral domain and π an element of R such that the ideal $\mathfrak{p} = \pi R$ is a smooth height one prime ideal of R, i.e., such that the localization $R_\mathfrak{p}$ is a discrete valuation ring. Let F be the fraction field of R and $R_\pi := R[1/\pi]$. Then there exists a bijection between*

(i) finitely generated torsion free R-modules M such that $M/\pi M$ is torsion free as well and

(ii) diagrams

$$\begin{array}{ccc} & & M_\pi \\ & & \downarrow{\scriptstyle\alpha} \\ M_\mathfrak{p} & \xrightarrow{\beta} & M_F, \end{array}$$

where M_π, M_F and $M_\mathfrak{p}$ are finitely generated modules over R_π, F and $R_\mathfrak{p}$, respectively,

$$M_F \cong M_\pi \otimes_{R_\pi} F \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} F$$

and the maps α and β are the maps that arise from the above isomorphisms by composition with the canonical morphisms $M_\pi \to M_\pi \otimes_{R_\pi} F$ and $M_\mathfrak{p} \to M_\mathfrak{p} \otimes_{R_\mathfrak{p}} F$, respectively.

The bijection is the obvious one, given by mapping M to the diagram

$$\begin{array}{ccc} & & M \otimes_R R_\pi \\ & & \downarrow{\scriptstyle\alpha} \\ M \otimes_R R_\mathfrak{p} & \xrightarrow{\beta} & M \otimes_R F. \end{array}$$

The inverse is obtained by mapping a diagram to $\mathrm{Im}(\alpha) \cap \mathrm{Im}\,\beta \subset M_F$, which turns out to be a finitely generated torsion free R-module which remains torsion free after tensoring with $R/(\pi)$ over R.

Based on this, the following is shown in [13], Prop. 1.13 ii):

Proposition 11.17 *Suppose $j : U = \operatorname{Spec} \mathbb{K} \to X = \operatorname{Spec} \mathbb{V}$ is corresponds to the generalization map $\mathbb{V} \to \mathbb{K}$ of the discrete valuation ring $\mathbb{V}$. Suppose τ is injective. Then $j_\#\underline{\mathcal{F}}$ is coherent. Furthermore, if $\mathcal{F}$ is locally free, then $j_\#\underline{\mathcal{F}}$ is locally free of the same rank.*

For the proof, one applies the above lemma to $R := \mathbb{V} \otimes A$ and π a uniformizer of $\mathbb{V}$. One is given $\underline{\mathcal{F}}$ on $\operatorname{Spec} R_\pi$ and hence on $\operatorname{Spec} F$, $F = \operatorname{Frac}(R)$, and one needs to perform a calculation on the discrete valuation ring $R_{\mathfrak{p}} \subset F$. Now any finitely generated F vector M_F spaces arises from an $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ by base change. Furthermore a morphism $(\sigma \times \mathrm{id})^* M_F \to M_F$ clearly arises from a morphism

$$(\pi^{nq})(\sigma \times \mathrm{id})^* M_{\mathfrak{p}} \cong (\sigma \times \mathrm{id})^*(\pi^n M_{\mathfrak{p}}) \to (\pi^n M_{\mathfrak{p}})$$

for n sufficiently large. Thus by the lemma one can find an extension. The union of any two extensions is again an extension. By an inductive argument, one can find an extension $\underline{\mathcal{M}}$ for which the cokernel of τ has minimal length over $R_{\mathfrak{p}}$. This extension will be the maximal one.

Proof of Theorem 11.15: Let $z_1, \ldots, z_r$ denote the codimension one points in Z. Let $\underline{\mathcal{F}}_l$, j_l be defined as in the previous corollary. By the above corollary, it suffices to show that for each l the τ-sheaf $j_{l\#}\underline{\mathcal{F}}_l$ is coherent. This follows from the previous proposition. ∎

Proof of Proposition 11.9: As in the previous proof it suffices to prove the proposition in the case where $X = \operatorname{Spec} R$, where R is a discrete valuation ring with fraction field F, and where $U = \operatorname{Spec} F$. This is treated in [13], Prop. 1.13 i). ∎

Assume that U is dense in X and that $j_\#\underline{\mathcal{F}}$ is coherent. We then have the short exact sequence

$$0 \longrightarrow j_!\underline{\mathcal{F}} \longrightarrow j_\#\underline{\mathcal{F}} \longrightarrow i_* i^* j_\#\underline{\mathcal{F}} \longrightarrow 0 \tag{51}$$

in $\mathbf{Crys}(X, A)$. The sheaf $i^* j_\#\underline{\mathcal{F}}$ is computable in $\mathbf{Coh}_\tau(X, A)$ and the following proposition is a useful tool for its computation.

Proposition 11.18 *Suppose X is regular and $\underline{\mathcal{F}}$ is torsion free. Define $f : \hat{X}_Z \to X$ to be the completion of the scheme X along Z and consider the pair of pullback diagrams*

$$\begin{tikzcd}
U \arrow[r, "j"] & X & Z \arrow[l, "i"'] \\
\hat{X}_Z \setminus Z \arrow[u, "f'"] \arrow[r, "\hat{j}"'] & \hat{X}_Z \arrow[u, "f"] & Z. \arrow[l, "\hat{i}"] \arrow[u, equal]
\end{tikzcd}$$

Define $\widehat{\underline{\mathcal{F}}} := f'^* \underline{\mathcal{F}}$. Then $f^* j_\#\underline{\mathcal{F}} \cong \hat{j}_\#\widehat{\underline{\mathcal{F}}}$ and in particular $i^* j_\#\underline{\mathcal{F}} \cong \hat{i}^* \hat{j}_\#\widehat{\underline{\mathcal{F}}}$.

Proof: By the above proposition, we may assume that $X = \operatorname{Spec} R$ is affine and Z is irreducible, so that Z is the closure of a single prime $\mathfrak{p} \in \operatorname{Spec} R$. If $\operatorname{codim}_X Z \geq 2$, then $j_\# = j_*$ and similarly $\hat{j}_\# = \hat{j}_*$, and so the assertion follows from well-known results in ring theory. Therefore we assume that $\mathfrak{p}$ has height 1. Denote by $\hat{R}$ the completion of R at $\mathfrak{p} \subset R$.

Clearly one has $f^* j_\#\underline{\mathcal{F}} \subset \hat{j}_\#\widehat{\underline{\mathcal{F}}}$. To prove the opposite inclusion, let $\widehat{\underline{\mathcal{G}}}$ be a coherent τ-subsheaf of $\hat{j}_\#\widehat{\underline{\mathcal{F}}}$ which contains $f^* j_\#\underline{\mathcal{F}}$. Using the normality of $\hat{X}_Z \otimes A$ as in the proof of Lemma 11.13, one can find an element $0 \neq g \in \mathfrak{p}$ such that $g\widehat{\underline{\mathcal{G}}} \subset f^* j_\#\underline{\mathcal{F}}$. (This is a based on a simple argument where one compares finitely generated modules for the discrete valuation rings $(R \otimes A)_{\mathfrak{p} \otimes A}$

and $(\hat{R}\otimes A)_{\mathfrak{p}\otimes A}$, which are the obvious localizations of the rings $R\otimes A$ and $\hat{R}\otimes A$.) This implies that $f_*\widehat{\mathcal{G}}\cap j_*\mathcal{F}\subset(1/g)j_\#\mathcal{F}$ is coherent. Hence we have $f_*\underline{\widehat{\mathcal{G}}}\cap j_*\underline{\mathcal{F}}\subset j_\#\underline{\mathcal{F}}$, and thus also $f_*\hat{j}_\#\underline{\widehat{\mathcal{F}}}\cap j_*\underline{\mathcal{F}}\subset j_\#\underline{\mathcal{F}}$.

It easily follows that the localization of $(j_\#\underline{\mathcal{F}})\otimes_R\hat{R}$ at $\mathfrak{p}$ agrees with that of $\hat{j}_\#\underline{\widehat{\mathcal{F}}}$ at $\mathfrak{p}$, because $(\hat{R}\otimes A)_{\mathfrak{p}\otimes A}$ is a discrete valuation ring. Since all other height one primes $\mathfrak{q}$ of $\operatorname{Spec}\hat{R}\otimes A$ are in $f'^{-1}(U\times\operatorname{Spec}A)$, it follows that $f^*(j_\#\underline{\mathcal{F}})$ agrees with $\hat{j}_\#\underline{\widehat{\mathcal{F}}}$ at any height one prime of $\hat{X}_Z\times\operatorname{Spec}A$. Both sheaves are torsion free and by normality of $\hat{R}\otimes A$, they must agree. ∎

11.2 The maximal extension of $\underline{\mathcal{F}}_{\mathcal{K}}$

Recall that $j_{\mathcal{K}}\colon\mathfrak{M}_{\mathcal{K}}\to\overline{\mathfrak{M}}_{\mathcal{K}}$ denotes the inclusion of the moduli scheme $\mathfrak{M}_{\mathcal{K}}$ into the compactification $\overline{\mathfrak{M}}_{\mathcal{K}}$ constructed by Drinfeld. The τ-sheaves $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ satisfy the hypothesis of Theorem 11.15. Therefore $j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ is a coherent τ-sheaf on $\overline{\mathfrak{M}}_{\mathcal{K}}$.

Let $i_{\mathcal{K}}:\mathfrak{M}_{\mathcal{K}}^{\infty}\hookrightarrow\overline{\mathfrak{M}}_{\mathcal{K}}$ denote the closed immersion of the cusps into $\overline{\mathfrak{M}}_{\mathcal{K}}$, which was constructed in Section 2. The central result on $j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ is the following theorem whose proof will occupy the remainder of this subsection.

Theorem 11.19 *For each connected component c of $\mathfrak{M}_{\mathcal{K}}^{\infty}$, there exists a unique A-module P_c of rank one such that there is a nil-isomorphism*

$$\bigoplus_c\mathbb{1}_{c,A}\otimes_A P_c\longrightarrow i_{\mathcal{K}}^*j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}},\tag{52}$$

where the summation is over all components c. If one defines

$$\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\infty}:=\bigoplus_c\mathbb{1}_{c,A}\otimes_A P_c^{\otimes n},$$

where the summation is again over all components c of $\mathfrak{M}_{\mathcal{K}}^{\infty}$, then for all $n\in\mathbb{N}_0$ there is a nil-isomorphism

$$\underline{\mathcal{F}}_{\mathcal{K}}^{(n),\infty}\longrightarrow i_{\mathcal{K}}^*j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$$

induced from (52) by taking symmetric powers. In particular, in $\mathbf{Crys}(\overline{\mathfrak{M}}_{\mathcal{K}},A)$ *one has the induced short exact sequence*

$$0\longrightarrow j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}\longrightarrow j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}\longrightarrow i_{\mathcal{K}*}\Big(\bigoplus_c\mathbb{1}_{c,A}\otimes_A P_c^{\otimes n}\Big)\longrightarrow 0.\tag{53}$$

Let us observe that for the proof of the theorem, we may apply Proposition 11.18. Thus it suffices to consider the local situation $i'_{\mathcal{K}}:\mathfrak{M}_{\mathcal{K}}^{\infty}\hookrightarrow\widehat{\mathfrak{M}}_{\mathcal{K}}$. In the discussion leading to Proposition 2.14, we obtained an explicit description for the spaces involved, and an analytic description of the universal Drinfeld module on the generic fiber $\widetilde{\mathfrak{M}}_{\mathcal{K}}$ of $\widehat{\mathfrak{M}}_{\mathcal{K}}$. The corresponding open immersion is denoted by $j'_{\mathcal{K}}$.

We first consider this situation in the case $\mathcal{K}=\mathcal{K}(\mathfrak{n})$, where $\mathfrak{n}$ is a fixed proper non-zero principal ideal of A. The general situation is discussed toward the end of this subsection. By Lemma 2.11, the scheme $\widehat{\mathfrak{M}}_{\mathcal{K}}$ of infinitesimal neighborhoods of the cusps is a disjoint union of the corresponding schemes at the individual cusps. Therefore it suffices to consider the following situation described in Subsection 2.2:

We denote by $\mathcal{R}_{\mathfrak{n}}$ the coordinate ring of the affine scheme $\mathfrak{M}_{\mathfrak{n}}^1$ and by $\mathcal{Q}_{\mathfrak{n}}$ its fraction field. $\mathcal{R}_{\mathfrak{n}}$ is a regular ring of dimension one and of finite type over k. If π denotes an indeterminate, then we had obtained identifications $\mathfrak{M}_{\mathfrak{n}}^{\infty} \cong \operatorname{Spec} \mathcal{R}_{\mathfrak{n}}$, $\widehat{\mathfrak{M}}_{\mathfrak{n}} \cong \operatorname{Spec} \mathcal{R}_{\mathfrak{n}}[[\pi]]$ and $\widetilde{\mathfrak{M}}_{\mathfrak{n}} \cong \operatorname{Spec} \mathcal{R}_{\mathfrak{n}}((\pi))$ in such a way that $i'_{\mathcal{K}}$, when restricted to an individual cusp, corresponds to the canonical map $\mathcal{R}_{\mathfrak{n}}[[\pi]] \twoheadrightarrow \mathcal{R}_{\mathfrak{n}}$ and $j'_{\mathcal{K}}$ to inverting π in $\mathcal{R}_{\mathfrak{n}}[[\pi]]$. Let $\mathrm{v}_{\mathfrak{n}}$ denote the valuation on $\mathcal{Q}_{\mathfrak{n}}((\pi))$ with $\mathrm{v}_{\mathfrak{n}}(\pi) = 1$. For $\mathfrak{p} \in \operatorname{Max} \mathcal{R}_{\mathfrak{n}}$, define $k_{\mathfrak{p}}$ as $\mathcal{R}_{\mathfrak{n}}/\mathfrak{p}$ and $\mathrm{v}_{\mathfrak{p}}$ as the valuation on $k_{\mathfrak{p}}((\pi))$ such that $\mathrm{v}_{\mathfrak{p}}(\pi) = 1$.

Our main tool to establish the above theorem is to pass from algebraic τ-sheaves $\underline{\mathcal{F}}$ to A-analytic τ-sheaves $\underline{\mathcal{F}}^{A\text{-an}}$. If $X = \operatorname{Spec} \mathbb{V}$ where $\mathbb{V}$ is a discrete valuation ring, this procedure was considered in [13], Ch. 1. Because the base X is not rigid analytic, we chose the term A-analytic instead of $\mathfrak{D}_A$-rigid. Our aim is to make a clear distinction between the analytic considerations in this subsection and in Section 8.

One can define in both, algebraic and A-analytic contexts a notion of maximal extension and the respective reductions to the special fibers agree. The advantage of the analytic set-up is that the maximal extension of $\underline{\mathcal{F}}^{A\text{-an}}$ does contain a subsheaf 'of good reduction', i.e., the 'good part' of the special fiber is visible over the generic fiber. In the situation at hand, this subsheaf will be the unit τ-sheaf $\mathbb{1}^{A\text{-an}}_{\mathfrak{M}^{\infty}_{\mathcal{K}},A}$.

From now on, we fix a finite flat map $k[T] \to A$ and a $k[T]$ basis $a_1, \ldots, a_d$ of A. We define $(\mathcal{R}_{\mathfrak{n}}[[\pi]] \otimes A)^{A\text{-an}} \subset (\mathcal{R}_{\mathfrak{n}}[[\pi]] \otimes A)\hat{\otimes}_{k[T]} k[[T]]$ as

$$\Big\{ \sum_{n=0}^{\infty} \sum_{i=1}^{d} r_{n,i} \otimes a_i T^n : r_{n,i} \in \mathcal{R}_{\mathfrak{n}}[[\pi]], \frac{1}{n} \mathrm{v}_{\mathfrak{n}}(r_{n,i}) \to \infty \text{ for } n \to \infty \Big\}.$$

This is clearly a ring under the obvious operations. It is independent of the chosen map $k[T] \to A$. Its reduction modulo π is isomorphic to $\mathcal{R}_{\mathfrak{n}} \otimes A$. Furthermore it is a subring of the ring $(\mathcal{R}_{\mathfrak{n}} \otimes A)[[\pi]]$ in an obvious way. One defines the ring $(\mathcal{R}_{\mathfrak{n}}((\pi)) \otimes A)^{A\text{-an}} := (\mathcal{R}_{\mathfrak{n}}[[\pi]] \otimes A)^{A\text{-an}}[1/\pi]$. Similar rings are also defined for $\mathcal{Q}_{\mathfrak{n}}$ and $k_{\mathfrak{p}}$, $\mathfrak{p} \in \operatorname{Max} \mathcal{R}_{\mathfrak{n}}$, instead of $\mathcal{R}_{\mathfrak{n}}$. All these rings are noetherian.

The following diagram exhibits the natural maps arising from specialization and generalization of schemes:

$$\begin{tikzcd}
(\mathcal{Q}_{\mathfrak{n}}((\pi)) \otimes A)^{A\text{-an}} & (\mathcal{R}_{\mathfrak{n}}((\pi)) \otimes A)^{A\text{-an}} \arrow[l, hook'] \arrow[r, two heads] & (k_{\mathfrak{p}}((\pi)) \otimes A)^{A\text{-an}} \\
(\mathcal{Q}_{\mathfrak{n}}[[\pi]] \otimes A)^{A\text{-an}} \arrow[u, hook] \arrow[d, two heads] & (\mathcal{R}_{\mathfrak{n}}[[\pi]] \otimes A)^{A\text{-an}} \arrow[l, hook'] \arrow[r, two heads] \arrow[u, hook] \arrow[d, two heads] & (k_{\mathfrak{p}}[[\pi]] \otimes A)^{A\text{-an}} \arrow[u, hook] \arrow[d, two heads] \\
\mathcal{Q}_{\mathfrak{n}} \otimes A & \mathcal{R}_{\mathfrak{n}} \otimes A \arrow[l, hook'] \arrow[r, two heads] & k_{\mathfrak{p}} \otimes A.
\end{tikzcd}$$

In the remainder of this subsection R will always denote one of the rings $\{\mathcal{Q}_{\mathfrak{n}}, \mathcal{R}_{\mathfrak{n}}, k_{\mathfrak{p}} : \mathfrak{p} \in \operatorname{Max} \mathcal{R}_{\mathfrak{n}}\}$ and $\mathcal{R}$ one of the rings $R[[\pi]]$, $R((\pi))$. The map $\sigma \times \mathrm{id}$ on $\mathcal{R} \otimes A$ has an obvious extension to $(\mathcal{R} \otimes A)^{A\text{-an}}$ which we again denote by $\sigma \times \mathrm{id}$. For a finitely generated τ-module (M, τ) on $\mathcal{R} \otimes A$, one defines $(M, \tau)^{A\text{-an}}$ as $(M, \tau) \otimes_{\mathcal{R} \otimes A} (\mathcal{R} \otimes A)^{A\text{-an}}$. The generalization map $\operatorname{Spec} R((\pi)) \to \operatorname{Spec} R[[\pi]]$ is denoted j_R and the specialization map $\operatorname{Spec} R \to \operatorname{Spec} R[[\pi]]$ by i_R.

An *A-analytic τ-sheaf* $\underline{\tilde{\mathcal{F}}} = (\tilde{\mathcal{F}}, \tau)$ *on* $\operatorname{Spec} \mathcal{R}$ consists of a coherent sheaf $\tilde{\mathcal{F}}$ on $\operatorname{Spec}(\mathcal{R} \otimes A)^{A\text{-an}}$ together with a morphism $(\sigma \times \mathrm{id})^* \tilde{\mathcal{F}} \xrightarrow{\tau} \tilde{\mathcal{F}}$. One defines the notion of *extension* and *maximal extension* for the morphism j_R. If a maximal extension of $\underline{\tilde{\mathcal{F}}}$ on $\operatorname{Spec} R((\pi))$ to $\operatorname{Spec} R[[\pi]]$ exists, it is denoted $\underline{\tilde{\mathcal{F}}}^{\max}$.

A locally free analytic τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\operatorname{Spec} R[[\pi]]$ with injective τ is said to have *good reduction* if the map τ on the locally free τ-sheaf $i_R^* \underline{\tilde{\mathcal{F}}}$ on $\operatorname{Spec} R$ over A is injective. Note that if R is a field, then any locally free τ-sheaf $\overline{\underline{\mathcal{F}}}$ on

$\operatorname{Spec} R \otimes A$ contains a locally free τ-subsheaf $\overline{\underline{\mathcal{G}}}$ such that $\overline{\underline{\mathcal{G}}} \to \overline{\underline{\mathcal{F}}}$ is an injective nil-isomorphism and $\tau_{\overline{\underline{\mathcal{G}}}}$ is a monomorphism.

The following result is due to Gardeyn, [13], Ch. 1. Parts a) and c) are based on an A-analytic analogue of Lemma 11.16 and on a comparison of the two versions of this lemma.

Proposition 11.20 *Let $\tilde{\underline{\mathcal{F}}}$ be an A-analytic torsion free τ-sheaf on* $\operatorname{Spec} R((\pi))$.

a) Assume that $\tau_{\tilde{\underline{\mathcal{F}}}}$ is a monomorphism and that $\tilde{\underline{\mathcal{F}}}$ has an extension $\tilde{\underline{\mathcal{G}}}$ to $\operatorname{Spec} R[[\pi]]$. *Then $\tilde{\underline{\mathcal{F}}}$ has a torsion free maximal extension, denoted $\tilde{\underline{\mathcal{F}}}^{\max}$. Its reduction $i_R^*(\tilde{\underline{\mathcal{F}}}^{\max})$ is torsion free as well. If R is a field and $\tilde{\underline{\mathcal{G}}}$ is locally free, then $\tilde{\underline{\mathcal{F}}}^{\max}$ is locally free.*

b) Suppose R is a finite field and $\tilde{\underline{\mathcal{G}}}$ is a maximal locally free extension of $\tilde{\underline{\mathcal{F}}}$. Then there exist

(i) a finite set of points S of $\operatorname{Spec} R \otimes A$,

(ii) a locally free analytic τ-subsheaf $\tilde{\underline{\mathcal{G}}}'$ of $\tilde{\underline{\mathcal{G}}}$ on the formal scheme $X :=$ $\operatorname{Spm}(R[[\pi]] \otimes A)^{A\text{-an}} \smallsetminus S$

such that $\tilde{\underline{\mathcal{G}}}'$ has good reduction and the induced map $i_R^*\tilde{\underline{\mathcal{G}}}' \hookrightarrow i_R^*\tilde{\underline{\mathcal{G}}}$ is an injective nil-isomorphism on $\operatorname{Spec}(R \otimes A) \smallsetminus S$.

c) Suppose that $\tilde{\underline{\mathcal{F}}} = \underline{\mathcal{F}}^{A\text{-an}}$ for some torsion free, coherent τ-sheaf on $\operatorname{Spec} R((\pi))$ *such that $\tau_{\mathcal{F}}$ is a monomorphism. Then $j_{R\#}\underline{\mathcal{F}}$ is coherent, $\tilde{\underline{\mathcal{F}}}^{\max}$ exists, and there is a canonical isomorphism $(j_{R\#}\underline{\mathcal{F}})^{A\text{-an}} \to \tilde{\underline{\mathcal{F}}}^{\max}$. In particular one has $i_R^*(j_{R\#}\underline{\mathcal{F}}) \cong i_R^*(\tilde{\underline{\mathcal{F}}}^{\max})$ on $R \otimes A$.*

Note that unlike in the algebraic situation, part a) needs the existence of an extension of $\tilde{\underline{\mathcal{F}}}$ to $R[[\pi]]$.

PROOF: In the case where R is a field, the results as stated are from [13], Prop. 1.15 ii), iii), Thm. 1.21. The case where $R = \mathcal{R}_{\mathfrak{n}}$ can be obtained along the same lines as the results quoted, and we leave it to the reader to fill in the details. An important step in the proof is to generalize [13], Lem. 1.3, to the case where the base has dimension 2. This step is based on Lemma 11.16. ∎

In analogy to Corollary 11.10, one has the following result. The proof is similar, again based on [13], Ch. 1, and thus omitted.

Proposition 11.21 *Suppose $\mathcal{R} = \mathcal{R}_{\mathfrak{n}}[[\pi]]$ and $\mathcal{S} \in \{\mathcal{R}_{\mathfrak{n}}((\pi)), \mathcal{Q}_{\mathfrak{n}}[[\pi]], \mathcal{Q}_{\mathfrak{n}}((\pi))\}$. Then $\underline{\mathbb{1}}^{A\text{-an}}_{\operatorname{Spec}\mathcal{R},A}$ is the maximal extension of $\underline{\mathbb{1}}^{A\text{-an}}_{\operatorname{Spec}\mathcal{S},A}$ to* $\operatorname{Spec}\mathcal{R}$. *In particular if $\tilde{\underline{\mathcal{F}}}$ is an A-analytic τ-sheaf on* $\operatorname{Spec}\mathcal{R}$ *such that $\underline{\mathbb{1}}^{A\text{-an}}_{\operatorname{Spec}\mathcal{S},A}$ is contained in $\tilde{\underline{\mathcal{F}}} \otimes_{\mathcal{R}} \mathcal{S}$, then*

$$\underline{\mathbb{1}}^{A\text{-an}}_{\operatorname{Spec}\mathcal{S},A} \cap \tilde{\underline{\mathcal{F}}} \subset \underline{\mathbb{1}}^{A\text{-an}}_{\operatorname{Spec}\mathcal{R},A} \subset \tilde{\underline{\mathcal{F}}} \otimes_{\mathcal{R}} \mathcal{S}.$$

We now come to the basic construction needed for the proof of Theorem 11.19. For this recall that in Section 2, we had defined a universal rank 1 Drinfeld-module $\varphi^1_{\mathfrak{n}}$ on $\mathcal{R}_{\mathfrak{n}}$. Its pullback along the zero section of $\mathcal{R}_{\mathfrak{n}}[[\pi]] \to \mathcal{R}_{\mathfrak{n}}$ defines a rank 1 Drinfeld module, which we denote by φ, and the pullback of φ to $\mathcal{R}_{\mathfrak{n}}((\pi))$ is again denoted φ. On $\mathcal{R}_{\mathfrak{n}}((\pi))$, we also had defined a rank 2 Drinfeld-module φ' with semi-stable reduction of rank 1, cf. Proposition 2.9. Furthermore, in loc. cit. we had defined an exponential map

$$e_\lambda \colon \mathcal{R}_{\mathfrak{n}}((\pi)) \to \mathcal{R}_{\mathfrak{n}}((\pi)) : x \mapsto \sum_{n=0}^{\infty} c_n x^{q^n}$$

with $c_n \in \pi^{e_n}\mathcal{R}_{\mathfrak{n}}$ where the e_n are positive integers with $\log_q e_n \to \infty$.

Define $\underline{\mathcal{M}}_2 := (M_2, \tau_2)$ as the τ-module on $\mathcal{R}_{\mathfrak{n}}((\pi))$ corresponding to φ'. I.e. $M_2 = R((\pi))\{\tau\}$ where $\mathcal{R}_{\mathfrak{n}}((\pi))$ acts via left multiplication and $a \in A$ acts via composition with φ'_a on the right. Furthermore, define $\underline{\mathcal{M}}_1 := (M_1, \tau_1)$ as the τ-module on $\mathcal{R}_{\mathfrak{n}}((\pi))$ corresponding to φ. The modules M_2 and M_1 are projective over $\mathcal{R}_{\mathfrak{n}}((\pi)) \otimes A$ of ranks 2 and 1, respectively. The module M_2 is free over $\mathcal{R}_{\mathfrak{n}}((\pi)) \otimes k[T]$ with basis $m_1 = 1, m_2 = \tau, \ldots, m_{2d} = \tau^{2d-1}$ and similarly M_1 is free with basis $m'_1 = 1, m'_2 = \tau, \ldots, m'_d = \tau^{d-1}$. With respect to the latter basis, the map τ on M_1 is given as $\tau = \alpha(\sigma \times \mathrm{id})$ for some $d \times d$-matrix

$$\alpha = \begin{pmatrix} 0 & 0 & \ldots & 0 & -r_0/r_d \\ 1 & 0 & \ldots & 0 & -r_1/r_d \\ 0 & 1 & \ldots & 0 & -r_2/r_d \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & -r_{d-1}/r_d \end{pmatrix} + T \begin{pmatrix} 0 & \ldots & 0 & 1/r_d \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 \end{pmatrix}$$

where the $r_i \in \mathcal{R}_{\mathfrak{n}}$ and $r_d \in \mathcal{R}_{\mathfrak{n}}^*$, because φ arises via base change from $\varphi^1_{\mathfrak{n}}$.

Lemma 11.22 *There is a well-defined map of A-analytic τ-sheaves*

$$\mu \colon \underline{\mathcal{M}}_2^{A\text{-an}} \to \underline{\mathcal{M}}_1^{A\text{-an}}$$

defined by mapping the basis element m_l, $1 \le l \le 2d$ to

$$\sum_{n \ge 0} c_n^{(q^{l-1})} \alpha \alpha^{(q)} \ldots \alpha^{(q^{n+l-2})} \begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

with respect to the basis $m'_1, \ldots, m'_d$.

Proof: Having the above explicit description of α and the condition $\log_q e_n \to \infty$, it follows easily that the sums defining the images of the m_i are column vectors of length d defined over $(\mathcal{R}_{\mathfrak{n}}[[\pi]] \otimes k[T])^{k[T]\text{-an}}$. Thus μ is well-defined. It clearly gives rise to a map of A-analytic τ-sheaves on $R((\pi))$. ∎

We define $\tilde{\underline{\mathcal{M}}}_2 := \underline{\mathcal{M}}_2^{A\text{-an}}$, $\tilde{\underline{\mathcal{M}}}_1 := \underline{\mathcal{M}}_1^{A\text{-an}}$ and $\tilde{\underline{\mathcal{H}}} := \ker \mu$.

Lemma 11.23 *The map μ is surjective and $\tilde{\underline{\mathcal{H}}}$ a locally free A-analytic τ-sheaf on which τ is an isomorphism.*

Proof: For each $\mathfrak{p} \in \operatorname{Max} \mathcal{R}_{\mathfrak{n}}$ there is a specialization map $\mu_{\mathfrak{p}}$ for the fiber at $\mathfrak{p}$. By [13], Thm. 6.22, the resulting map $\mu_{\mathfrak{p}}$ is surjective and its kernel is isomorphic to $\mathbb{1}^{A\text{-an}}_{k_{\mathfrak{p}}((\pi)),A}$. Since this holds for all $\mathfrak{p}$, the map μ is surjective, $\tilde{\underline{\mathcal{H}}}$ is locally free of rank one over $\operatorname{Spec}(R((\pi)) \otimes A)^{A\text{-an}}$ and $\tau_{\tilde{\underline{\mathcal{H}}}}$ is surjective. As any surjective map of locally free sheaves of the same rank is an isomorphism, so is $\tau_{\tilde{\underline{\mathcal{H}}}}$, and thus we have proved all assertions of the lemma. ∎

Lemma 11.24 *The sheaf $\tilde{\underline{\mathcal{H}}}$ is free over* $\operatorname{Spec}(\mathcal{R}_{\mathfrak{n}}((\pi)) \otimes k[T])^{k[T]\text{-an}}$ *of rank d.*

Proof: As in the proof of [13], Thm. 6.22, one can construct a sequence of $k[T]$-analytic τ-sheaves $\tilde{\underline{\mathcal{G}}}_i$ on $R((\pi))$ with surjective morphism $\mu_i \colon \tilde{\underline{\mathcal{G}}}_{i-1} \to \tilde{\underline{\mathcal{G}}}_i$, $i = 1, \ldots, d$ such that $\tilde{\underline{\mathcal{G}}}_0 \cong \tilde{\underline{\mathcal{M}}}_2$ and $\tilde{\underline{\mathcal{G}}}_d \cong \tilde{\underline{\mathcal{M}}}_1$ and such that the $\tilde{\underline{\mathcal{G}}}_i$ correspond to Drinfeld-$k[T]$-modules of rank $2d-i$ on $R((\pi))$. (The minimal basis needed in loc. cit. can be obtained over $\mathcal{Q}_{\mathfrak{n}}((\pi))$; the construction still works over $\mathcal{R}_{\mathfrak{n}}((\pi))$.)

As the $\underline{\tilde{\mathcal{G}}}_i$ are attached to Drinfeld-modules, the underlying sheaves are free modules over $(\mathcal{R}_{\mathfrak{n}}((\pi)) \otimes k[T])^{k[T]-\mathrm{an}}$ of rank $2d-i$. Let $\underline{\tilde{\mathcal{H}}}_i$ denote the kernel of μ_i. Taking the determinant in the short exact sequence

$$0 \longrightarrow \underline{\tilde{\mathcal{H}}}_i \longrightarrow \underline{\tilde{\mathcal{G}}}_{i-1} \xrightarrow{\mu_i} \underline{\tilde{\mathcal{G}}}_i \longrightarrow 0$$

yields that $\underline{\tilde{\mathcal{H}}}_i$ is free over $(\mathcal{R}_{\mathfrak{n}}((\pi)) \otimes k[T])^{k[T]-\mathrm{an}}$ of rank one. Thus $\underline{\tilde{\mathcal{H}}}$ has a d-step filtration all of whose subquotients are free of rank one. It follows that $\underline{\tilde{\mathcal{H}}}$ itself is free of rank d. ∎

Proposition 11.25 *The sheaf $\underline{\tilde{\mathcal{H}}}$ is isomorphic to $(\mathbb{1}_{\operatorname{Spec} R((\pi)),A})^{A\text{-an}} \otimes_A P$ where P is a projective rank one module over A.*

PROOF: We claim that over $\operatorname{Spec}(R((\pi)))[T])^{k[T]\text{-an}}$ the sheaf $\underline{\tilde{\mathcal{H}}}$ is isomorphic to $((\mathbb{1}_{\operatorname{Spec} R((\pi)),\mathbb{F}_q[T]})^{k[T]\text{-an}})^d$. Assuming the claim for the moment, it follows that the set of τ-invariants $P := \underline{\tilde{\mathcal{H}}}^\tau$ is a free $k[T]$-module of rank d. But it also is a module over A, and hence it must be a projective rank one A-module. It follows that $P \otimes_A (\mathbb{1}_{R((\pi)),A})^{A\text{-an}} \to \underline{\tilde{\mathcal{H}}}$ is well-defined. Furthermore as a map of $k[T]$-analytic τ-sheaves it is an isomorphism and hence the proposition follows.

To prove the claim, let the modules $\underline{\tilde{\mathcal{H}}}_i$ be defined as in the previous proof. We will first show that each $\underline{\tilde{\mathcal{H}}}_i$ is isomorphic to $(\mathbb{1}_{\mathcal{R}_{\mathfrak{n}}((\pi)),k[T]})^{k[T]\text{-an}}$. By [13], Thm. 6.22, we know that after base change to $\operatorname{Spec}(\mathcal{Q}_{\mathfrak{n}}((\pi))[T])^{k[T]-\mathrm{an}}$ the sheaf $\underline{\tilde{\mathcal{H}}}_i$ becomes isomorphic to $(\mathbb{1}_{\mathcal{Q}_{\mathfrak{n}}((\pi)),k[T]})^{A\text{-an}}$. By Proposition 11.21, it follows that $\underline{\tilde{\mathcal{H}}}_i$ must be a $k[T]$-analytic τ subsheaf of $(\mathbb{1}_{\mathcal{R}_{\mathfrak{n}}((\pi)),k[T]})^{A\text{-an}}$.

Because $\tilde{\mathcal{H}}_i$ is free, there exists $f \in (\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}}$ such that $\underline{\mathcal{H}}_i = f(\mathbb{1}_{\mathcal{R}_{\mathfrak{n}}((\pi)),k[T]})^{k[T]\text{-an}}$. Because τ is surjective, f must satisfy the equation $f = uf^{(q)}$ for some unit $u \in (\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}}$. To analyze this equation we pass to the ring $(\mathcal{R}_{\mathfrak{n}}[T])((\pi))$ which sits in the pullback diagram of rings

$$\begin{array}{ccc} (\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}} & \subset & (\mathcal{R}_{\mathfrak{n}}[T])((\pi)) \\ \cap & & \cap \\ (\mathcal{Q}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}} & \subset & (\mathcal{Q}_{\mathfrak{n}}[T])((\pi)). \end{array} \tag{54}$$

An explicit calculation shows that over $(\mathcal{R}_{\mathfrak{n}}[T])((\pi))$ we may write $f = u_1\pi^{n_1}y$ for some unit u_1 of $(\mathcal{R}_{\mathfrak{n}}[T])[[\pi]]$, some $n_1 \in \mathbb{Z}$ and some $0 \neq y \in k[T]$. Because $\underline{\tilde{\mathcal{H}}}_i$ becomes isomorphic to $(\mathbb{1}_{\mathcal{Q}_{\mathfrak{n}}((\pi)),k[T]})^{k[T]\text{-an}}$ over the ring $(\mathcal{Q}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}}$ one concludes that f^{-1} lies in $(\mathcal{Q}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}}$ which implies $y \in k^*$. But then f^{-1} is an element of $(\mathcal{R}_{\mathfrak{n}}[T])((\pi))$ as well and the above pullback shows that f is a unit in $(\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}}$. Hence $\underline{\tilde{\mathcal{H}}}_i$ is isomorphic to $(\mathbb{1}_{\mathcal{R}_{\mathfrak{n}}((\pi)),k[T]})^{k[T]\text{-an}}$, as asserted.

By the above we know that $\underline{\tilde{\mathcal{H}}}$ can be written with respect to a suitable basis as

$$\underline{\tilde{\mathcal{H}}} = (((\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{A\text{-an}})^d, \alpha(\sigma \times \mathrm{id})), \alpha = \begin{pmatrix} 1 & x_{12} & x_{13} & \dots & x_{1d} \\ 0 & 1 & x_{23} & \dots & x_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

for certain $x_{ij} \in (\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{A\text{-an}}$.

Let us consider the special case $d = 2$. The general case will follow by a straightforward induction procedure and its proof is left to the reader. We claim that the set of τ-invariants over $(\mathcal{R}_{\mathfrak{n}}[T])((\pi))$ is the same as that over

$(\mathcal{Q}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}}$. If this is shown, the set of τ-invariants of $\underline{\tilde{\mathcal{H}}}$ is free over $k[T]$ of rank 2. Correspondingly one has a monomorphism

$$((\mathbb{1}_{\mathcal{R}_{\mathfrak{n}}((\pi)),k[T]})^{k[T]\text{-an}})^2 \hookrightarrow \underline{\tilde{\mathcal{H}}},$$

and Proposition 11.21 shows that it must be an isomorphism.

To compute the τ-invariants, we consider again the pullback diagram (54). The equations for a τ-invariant vector $(a, b) \in (\mathcal{Q}_{\mathfrak{n}}[T])((\pi))^2$ are

$$a = a^{(q)} + x_{12}b^{(q)}, \quad b = b^{(q)}$$

with $x_{12} \in (\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{A\text{-an}}$. This means that $b \in k[T]$ and that a satisfies an equation

$$a^{(q)} - a = x$$

for some $x \in (\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{A\text{-an}}$. We consider the last equation over the ring $(\mathcal{R}_{\mathfrak{n}}[T])((\pi))$. Thus writing $a = \sum_{n\gg-\infty} \alpha_n\pi^n$ and $x = \sum_{n\gg-\infty} \xi_n\pi^n$ with $\alpha_n \in \mathcal{Q}_{\mathfrak{n}}[T]$ and $\xi_n \in \mathcal{R}_{\mathfrak{n}}[T]$ we have

$$\sum_n \alpha_n^{(q)}\pi^{qn} - \sum_n \alpha_n\pi^n = \sum_n \xi_n\pi^n.$$

Comparing coefficients of the π^n, one may break up the equation into an equation for all $n > 0$, an equation for $n = 0$ and an equation for the finitely many terms with $n < 0$, say n_0 is the smallest index of a non-zero α_n. Looking at the terms with $n > 0$, it easily follows that for $n > 0$ all α_n are in $\mathcal{R}_{\mathfrak{n}}[T]$. For $n = 0$ one has the equation

$$\alpha_0^{(q)} - \alpha_0 = \xi_0$$

in $\mathcal{Q}_{\mathfrak{n}}[T]$. This leads to Artin-Schreier equations for all the coefficients of α_0 with respect to T where the coefficients of ξ_0 are in $\mathcal{R}_{\mathfrak{n}}$. So the coefficients of α_0 are integral over $\mathcal{R}_{\mathfrak{n}}$ and lie in $\mathcal{Q}_{\mathfrak{n}}$. Because $\mathcal{R}_{\mathfrak{n}}$ is regular, hence normal, the coefficients must lie in $\mathcal{R}_{\mathfrak{n}}$, and hence $\alpha_0 \in \mathcal{R}_{\mathfrak{n}}[T]$.

Finally, for $n < 0$ one first considers the equation for α_{n_0}. It reads $\alpha_{n_0}^{(q)} = \xi_{qn_0}$. As in the case $n = 0$, one uses the normality of $\mathcal{R}_{\mathfrak{n}}$ and comparison of coefficients with respect to T to obtain $\alpha_{n_0} \in \mathcal{R}_{\mathfrak{n}}$. Upwards induction starting with n_0 now easily shows that all α_n must lie in $\mathcal{R}_{\mathfrak{n}}$. The pullback property of diagram (54) shows that $(a, b) \in (\mathcal{R}_{\mathfrak{n}}((\pi))[T])^{k[T]\text{-an}}$. This finishes the proof for $d = 2$. ∎

We have thus obtained the short exact sequence

$$0 \longrightarrow \underline{\tilde{\mathcal{H}}} \cong \mathbb{1}^{A\text{-an}}_{\mathcal{R}_{\mathfrak{n}}((\pi)),A} \otimes_A P \longrightarrow \underline{\tilde{\mathcal{M}}}_2 \longrightarrow \underline{\tilde{\mathcal{M}}}_1 \longrightarrow 0. \tag{55}$$

on $\operatorname{Spec}(R((\pi))\otimes A)^{A\text{-an}}$ for the A-analytic τ-sheaf $\underline{\tilde{\mathcal{M}}}_2$ attached to the universal algebraic τ-sheaf $\underline{\mathcal{F}}_{\mathfrak{n}}$ pulled back to the generic point $\operatorname{Spec}\mathcal{R}_{\mathfrak{n}}((\pi))$ of the formal neighborhood of the cusps.

We fix $n > 0$ and introduce the following sheaves related to $\underline{\mathcal{F}}_{\mathfrak{n}}^{(n-1)} := \operatorname{Sym}^{n-1}\underline{\mathcal{F}}_{\mathfrak{n}}$: Let $\underline{\tilde{\mathcal{G}}}_i$, $i = 0, \ldots, n$ be the canonical filtration of $\underline{\tilde{\mathcal{G}}}_n := \operatorname{Sym}^{n-1}\underline{\tilde{\mathcal{M}}}_2$. Then for $i = 1, \ldots, n$ the successive subquotients $\underline{\tilde{\mathcal{G}}}_i/\underline{\tilde{\mathcal{G}}}_{i-1}$ are isomorphic to $\underline{\tilde{\mathcal{H}}}^{\otimes(i-1)} \otimes (\underline{\tilde{\mathcal{M}}}_1)^{\otimes(n-i)}$. For $\mathfrak{p} \in \operatorname{Max}(\mathcal{R}_{\mathfrak{n}})$ let $\tilde{k}_{\mathfrak{p}} := \mathcal{R}_{\mathfrak{n}}/\mathfrak{p}$ and let $I_{\mathfrak{p}}$ denote the inertia subgroup of $\operatorname{Gal}(\tilde{k}_{\mathfrak{p}}((\pi))^{\text{sep}}/\tilde{k}_{\mathfrak{p}}((\pi)))$. For an A-analytic τ-sheaf $\underline{\tilde{\mathcal{F}}}$ on $\operatorname{Spec}\mathcal{R}_{\mathfrak{n}}$, $\mathcal{R}_{\mathfrak{n}}((\pi))$ or $\mathcal{R}_{\mathfrak{n}}[[\pi]]$, we denote by $\underline{\tilde{\mathcal{F}}}_{\bar{\mathfrak{p}}}$ its reduction under $\mathcal{R}_{\mathfrak{n}} \to \mathcal{R}_{\mathfrak{n}}/\mathfrak{p}$.

Using the results in [13], for each $\mathfrak{p} \in \operatorname{Spec}\mathcal{R}_{\mathfrak{n}}$, the sheaves $(\underline{\tilde{\mathcal{G}}}_{i,\bar{\mathfrak{p}}})^{\max}$ are well understood. They key assertion for the proof of Theorem 11.19 is that the natural morphism $(\underline{\tilde{\mathcal{G}}}_i^{\max})_{\bar{\mathfrak{p}}} \to (\underline{\tilde{\mathcal{G}}}_{i,\bar{\mathfrak{p}}})^{\max}$ is a nil-isomorphism, cf. Lemma 11.28. We will prove this by induction on i. The proof uses that the subquotients

$\underline{\tilde{\mathcal{G}}}_i/\underline{\tilde{\mathcal{G}}}_{i-1}$ arise from a τ-sheaf on $\mathcal{R}_\mathfrak{n}$, with everywhere good reduction, by base change to $\mathcal{R}_\mathfrak{n}((\pi))$.

Using that $\underline{\tilde{\mathcal{G}}}_n$ arises from the algebraic τ-sheaf $\operatorname{Sym}^{n-1}\underline{\mathcal{M}}_2$, one can easily prove the following by some inductive arguments (for the last assertion one uses that the action of $I_\mathfrak{p}$ on $\mathrm{Ta}_l(\underline{\tilde{\mathcal{M}}}_{2,\bar{\mathfrak{p}}})$ is unipotent and not potentially trivial):

Lemma 11.26 *For each i, the quotient $\underline{\tilde{\mathcal{G}}}_i/\underline{\tilde{\mathcal{G}}}_{i-1}$ is the A-analytic τ-sheaf attached to an algebraic τ-sheaf on $\operatorname{Spec}\mathcal{R}_\mathfrak{n}((\pi))$, which in turn arises by pullback from an algebraic τ-sheaf with everywhere good reduction defined on the scheme $\operatorname{Spec}\mathcal{R}_\mathfrak{n}$. In particular, it admits a maximal extension to $\operatorname{Spec}\mathcal{R}_\mathfrak{n}[[\pi]]$. Furthermore it is locally free.*

The sheaves $\underline{\tilde{\mathcal{G}}}_i$ admit a maximal extension to $\operatorname{Spec}\mathcal{R}_\mathfrak{n}[[\pi]]$ as well.

For all $l\in\operatorname{Spec}A$, $\mathfrak{p}\in\operatorname{Max}(\mathcal{R}_\mathfrak{n})$ and $i\in\{1,\ldots,n\}$ the invariants under $I_\mathfrak{p}$ of the Tate-modules $\mathrm{Ta}_l(\underline{\tilde{\mathcal{G}}}_{i,\bar{\mathfrak{p}}})$ are one-dimensional.

We now set up the notation for a more involved induction. Suppose we are given a short exact sequence of locally free A-analytic τ-sheaves on $\operatorname{Spec}\mathcal{R}_\mathfrak{n}((\pi))$

$$0\longrightarrow\underline{\tilde{\mathcal{M}}}'\longrightarrow\underline{\tilde{\mathcal{M}}}\longrightarrow\underline{\tilde{\mathcal{M}}}''\longrightarrow 0 \tag{56}$$

such that

(a) $\underline{\tilde{\mathcal{M}}}''$ is the A-analytic τ-sheaf attached to an algebraic τ-sheaf $\underline{\mathcal{M}}''$ which is obtained from a locally free algebraic τ-sheaf $\underline{\mathcal{M}}''_0$ on $\operatorname{Spec}\mathcal{R}_\mathfrak{n}$ with everywhere good reduction by pullback to $\operatorname{Spec}\mathcal{R}_\mathfrak{n}((\pi))$ along the canonical map $\mathcal{R}\to\mathcal{R}((\pi))$.

(b) $\underline{\tilde{\mathcal{M}}}$ has a maximal extension (to $\operatorname{Spec}\mathcal{R}_\mathfrak{n}[[\pi]]$).

(c) $\underline{\tilde{\mathcal{G}}}_1$ is contained in $\underline{\tilde{\mathcal{M}}}'$.

(d) For all $l\in\operatorname{Spec}A$ and $\mathfrak{p}\in\operatorname{Max}(\operatorname{Spec}\mathcal{R}_\mathfrak{n})$, the invariants under $I_\mathfrak{p}$ of the Tate-modules $\mathrm{Ta}_l(\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})$ are one-dimensional.

Using (a) and (b) yields the diagram

$$\begin{tikzcd}
 & & & {\underline{\tilde{\mathcal{M}}}''}^{\max} & \\
0 \arrow[r] & {\underline{\tilde{\mathcal{M}}}'}^{\max} \arrow[r] & \underline{\tilde{\mathcal{M}}}^{\max} \arrow[r] & \underline{\tilde{\mathcal{C}}} \arrow[r] \arrow[u] & 0,
\end{tikzcd} \tag{57}$$

where $\underline{\tilde{\mathcal{C}}}$ is defined as the cokernel of the map ${\underline{\tilde{\mathcal{M}}}'}^{\max}\to\underline{\tilde{\mathcal{M}}}^{\max}$. Because of the maximality of ${\underline{\tilde{\mathcal{M}}}'}^{\max}$, the right vertical map is an injection. Regarded as modules over $(\mathcal{R}_\mathfrak{n}[[\pi]]\otimes A)^{A\text{-an}}$, there are canonical monomorphisms from the objects of diagram (57) to those of diagram (56). We reduce the resulting diagram at $\mathfrak{p}$ and insert a middle row corresponding to the maximal extension $(\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max}$ to obtain the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}} \arrow[r] & \underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}} \arrow[r] & \underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}} \arrow[r] & 0 \\
 & & & (\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max} \arrow[u] & \\
0 \arrow[r] & (\underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}})^{\max} \arrow[r] \arrow[uu] & (\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max} \arrow[r] \arrow[uu] & \underline{\tilde{\mathcal{C}}}'_{\bar{\mathfrak{p}}} \arrow[r] \arrow[u] & 0 \\
 & ({\underline{\tilde{\mathcal{M}}}'}^{\max})_{\bar{\mathfrak{p}}} \arrow[r] \arrow[u] & (\underline{\tilde{\mathcal{M}}}^{\max})_{\bar{\mathfrak{p}}} \arrow[r] \arrow[u] & \underline{\tilde{\mathcal{C}}}_{\bar{\mathfrak{p}}} \arrow[r] \arrow[u] & 0,
\end{tikzcd} \tag{58}$$

where in the middle row $\underline{\tilde{\mathcal{C}}}'_{\bar{\mathfrak{p}}}$ is defined as the cokernel.

Lemma 11.27 *The natural morphism* $(\underline{\tilde{\mathcal{M}}}''^{\max})_{\bar{\mathfrak{p}}} \to (\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max}$ *is an isomorphism.*

PROOF: The A-analytic τ-sheaf $\underline{\tilde{\mathcal{M}}}''$ is obtained from $\underline{\mathcal{M}}''_0$ via pullback along $\mathcal{R} \to \mathcal{R}((\pi))$ and A-analytification. Let $f\colon \operatorname{Spec} \mathcal{R}[[\pi]] \to \operatorname{Spec} \mathcal{R}$ be the structure morphism and $f_{\bar{\mathfrak{p}}}$ its reduction modulo $\mathfrak{p}$. Because $\underline{\mathcal{M}}''_0$ has everywhere good reduction and is locally free, it follows that $(\underline{\tilde{\mathcal{M}}}'')^{\max} \cong (f^*\underline{\mathcal{M}}''_0)^{A\text{-an}}$ and $(\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max} \cong (f^*_{\bar{\mathfrak{p}}}\underline{\mathcal{M}}''_{0,\bar{\mathfrak{p}}})^{A\text{-an}}$. Because A-analytification commutes with specialization at $\mathfrak{p}$, the assertion follows. ∎

We make the following assumption on Diagram (58):

(e) The map $(\underline{\tilde{\mathcal{M}}}'^{\max})_{\bar{\mathfrak{p}}} \to (\underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}})^{\max}$ is a nil-isomorphism for all $\mathfrak{p}$.

Lemma 11.28 *Under the above assumptions, the following hold:*

(i) The map $(\underline{\tilde{\mathcal{M}}}^{\max})_{\bar{\mathfrak{p}}} \to (\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max}$ *is a nil-isomorphism for all* $\mathfrak{p}$.

(ii) $\underline{\tilde{\mathcal{C}}} \subset \pi(\underline{\tilde{\mathcal{M}}}'')^{\max}$, *so that* $\tau = 0$ *on* $\underline{\tilde{\mathcal{C}}}/(\pi)$.

We first prove the following

Sublemma 11.29 *Let I be any ideal of $\mathcal{R}_{\mathfrak{n}}[[\pi]]$ and $\underline{\tilde{\mathcal{N}}}' \hookrightarrow \underline{\tilde{\mathcal{N}}}$ a monomorphism of A-analytic τ-sheaves on $(\mathcal{R}_{\mathfrak{n}}[[\pi]] \otimes A)^{A\text{-an}}$. Then the kernel of the induced map $\underline{\tilde{\mathcal{N}}}'/I\underline{\tilde{\mathcal{N}}}' \hookrightarrow \underline{\tilde{\mathcal{N}}}/I\underline{\tilde{\mathcal{N}}}$ is nilpotent.*

PROOF: The proof is based on the Artin-Rees lemma and follows the argument given in [4], Prop. 3.1.8: The kernel in question is given $(I\underline{\tilde{\mathcal{N}}} \cap \underline{\tilde{\mathcal{N}}}')/I\underline{\tilde{\mathcal{N}}}'$. Because τ is $(\sigma \times \mathrm{id})$-linear, it follows that

$$\tau^n(I\underline{\tilde{\mathcal{N}}} \cap \underline{\tilde{\mathcal{N}}}') \subset I^{q^n}\underline{\tilde{\mathcal{N}}} \cap \underline{\tilde{\mathcal{N}}}' \overset{\text{Artin-Rees}}{\subset} I^{q^n-l}(I^l\underline{\tilde{\mathcal{N}}} \cap \underline{\tilde{\mathcal{N}}}') \subset I\underline{\tilde{\mathcal{N}}}'$$

for some fixed $l > 0$ and $n \gg 0$. ∎

We now prove the lemma:

PROOF: By the maximality of $(\underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}})^{\max}$ it easily follows that $\underline{\tilde{\mathcal{C}}}'_{\bar{\mathfrak{p}}}$ is torsion free and hence contained in $(\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max}$. For the same reason we have

$$\pi(\underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}})^{\max} = (\underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}})^{\max} \cap \pi(\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max},$$

so that $(\underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}})^{\max}/(\pi) \longrightarrow (\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max}/(\pi)$ is injective.

I) Our first claim is that $\underline{\tilde{\mathcal{C}}}'_{\bar{\mathfrak{p}}}$ is contained in $\pi(\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max}$: Assume otherwise. The reduction modulo π of the middle row yields the short exact sequence

$$0 \longrightarrow (\underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}})^{\max}/(\pi) \longrightarrow (\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max}/(\pi) \longrightarrow \underline{\tilde{\mathcal{C}}}'_{\bar{\mathfrak{p}}}/(\pi) \longrightarrow 0$$

on $\tilde{k}_{\mathfrak{p}} \otimes A$. By our assumption, the map $\underline{\tilde{\mathcal{C}}}'_{\bar{\mathfrak{p}}}/(\pi) \to (\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max}/(\pi)$, which we call α, has non-trivial image. Because $(\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max}/(\pi)$ is locally free and the corresponding τ is injective, τ is injective on the sheaf $\operatorname{Im}\alpha$ and $\operatorname{Im}\alpha$ is a non-trivial locally free sheaf over the Dedekind domain $\tilde{k}_{\mathfrak{p}} \otimes A$. We also know that $\underline{\tilde{\mathcal{G}}}_{1,\bar{\mathfrak{p}}} \subset (\underline{\tilde{\mathcal{M}}}'_{\bar{\mathfrak{p}}})^{\max}/(\pi)$, and it follows that $(\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max}/(\pi)$ contains a locally free subsheaf of rank at least two on which τ is injective.

Let $\overline{\underline{\mathcal{F}}}$ be the maximal locally free τ-subsheaf of $(\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max}/(\pi)$ on which τ is injective. Let $S \subset \operatorname{Spec} A$ be the finite set such that for $l \in \operatorname{Spec} A \smallsetminus S$ the morphism τ is injective on the fiber of $\overline{\underline{\mathcal{F}}}$ above l. We apply Proposition 11.20 b). Thus there exists a locally free analytic τ-subsheaf $\underline{\tilde{\mathcal{F}}}$ of $(\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})^{\max}$ of rank at least two on the formal scheme $X := \operatorname{Spm}(\tilde{k}_{\mathfrak{p}}[[\pi]] \otimes A)^{A\text{-an}} \smallsetminus S$, such that $\underline{\tilde{\mathcal{F}}}$ has reduction $\overline{\underline{\mathcal{F}}}$. In particular the Galois representation of $\operatorname{Gal}(\tilde{k}_{\mathfrak{p}}((\pi))^{\mathrm{sep}}/\tilde{k}_{\mathfrak{p}}((\pi)))$ on the Tate-module $\operatorname{Ta}_l(\underline{\tilde{\mathcal{F}}}) \subset \operatorname{Ta}_l(\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})$ is unramified for almost all l. For such l the submodule of $\operatorname{Ta}_l(\underline{\tilde{\mathcal{M}}}_{\bar{\mathfrak{p}}})$ of $I_{\mathfrak{p}}$-invariants has rank at least two, which contradics hypothesis (d).

II) We next prove part (ii): By the above, the map $\beta_{\bar{\mathfrak{p}}}$ defined as $\underline{\tilde{\mathcal{C}}}_{\bar{\mathfrak{p}}} \to (\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max}$ takes its image in $\pi(\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max}$. It follows that the induced map

$$\beta_{\bar{\mathfrak{p}}}/(\pi)\colon \underline{\tilde{\mathcal{C}}}_{\bar{\mathfrak{p}}}/(\pi) \to (\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max}/(\pi)$$

is zero. As this holds for all $\mathfrak{p} \in \operatorname{Max} \mathcal{R}_{\mathfrak{n}}$, Lemma 11.27 shows that the natural map

$$\beta_\pi\colon \underline{\tilde{\mathcal{C}}}/(\pi) \to \underline{\tilde{\mathcal{M}}}''^{\max}/(\pi)$$

is zero on all fibers $\mathfrak{p}$. The map β_π is a map of coherent algebraic τ-sheaves and hence it has nilpotent image in the locally free τ-sheaf $\underline{\tilde{\mathcal{M}}}''^{\max}/(\pi)$. However on the latter the map τ is injective, and hence β_π itself must be zero, which proves (ii).

III) We finally give the proof of part (i): Observe first that $(\mathcal{R}_{\mathfrak{n}}((\pi)) \otimes A)^{A\text{-an}}$ is equal to its subring $(\mathcal{R}_{\mathfrak{n}}[[\pi]] \otimes A)^{A\text{-an}}[1/\pi]$. It follows that we may find $m \in \mathbb{N}$ such that $\pi^m(\underline{\tilde{\mathcal{M}}}'')^{\max} \subset \underline{\tilde{\mathcal{C}}}$. Reducing mod $\mathfrak{p}$ yields

$$\pi^m(\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max} \subset \operatorname{Im}(\beta_{\bar{\mathfrak{p}}}) \subset \underline{\tilde{\mathcal{C}}}'_{\bar{\mathfrak{p}}} \subset \pi(\underline{\tilde{\mathcal{M}}}''_{\bar{\mathfrak{p}}})^{\max},$$

where we use I) for the inclusion on the right. Thus the inclusion morphism $\operatorname{Im}(\beta_{\bar{\mathfrak{p}}}) \subset \underline{\tilde{\mathcal{C}}}'_{\bar{\mathfrak{p}}}$ is a nil-isomorphism. Furthermore by the sublemma, the surjection $\underline{\tilde{\mathcal{C}}}_{\bar{\mathfrak{p}}} \to \operatorname{Im}(\beta_{\bar{\mathfrak{p}}})$ is a nil-isomorphism as well. We observed already that $(\underline{\tilde{\mathcal{M}}}'^{\max})_{\bar{\mathfrak{p}}} \to (\underline{\tilde{\mathcal{M}}}^{\max})_{\bar{\mathfrak{p}}}$ has a nilpotent kernel. Considering the middle and bottom sequences of diagram (58) together with the morphism between them and using our hypothesis on the left vertical map, it follows that the middle vertical map must be a nil-isomorphism. This is precisely assertion (i). ∎

Proof of Theorem 11.19: The proof of the theorem for $n = 0$ is an immediate consequence of Corollary 11.10. So from now one, we assume $n > 0$.

We first consider the case $\mathcal{K} = \mathcal{K}(\mathfrak{n})$ and $\mathfrak{n}$ principal. We apply Lemma 11.28 inductively to the sequences

$$0 \longrightarrow \underline{\tilde{\mathcal{G}}}_{i-1} \longrightarrow \underline{\tilde{\mathcal{G}}}_i \longrightarrow \underline{\tilde{\mathcal{G}}}_i/\underline{\tilde{\mathcal{G}}}_{i-1} \longrightarrow 0.$$

Lemma 11.26 and induction verify that the hypothesis (a)–(e) of the lemma are satisfied. It follows that for $i = 1, \dots, n$ we have exact sequences

$$0 \longrightarrow \underline{\tilde{\mathcal{G}}}_{i-1}^{\max} \longrightarrow \underline{\tilde{\mathcal{G}}}_i^{\max} \longrightarrow \underline{\tilde{\mathcal{C}}}_i \longrightarrow 0,$$

where the $\underline{\tilde{\mathcal{C}}}_i$ have nilpotent reductions modulo π for $i > 1$. Proposition 11.25 yields a unique projective A-module P of rank one such that $\underline{\tilde{\mathcal{G}}}_1^{\max} \cong \underline{\tilde{\mathcal{C}}}_1 \cong \underline{\tilde{\mathcal{H}}}^{\otimes(n-1)}$ has reduction $\underline{\mathbb{1}}_{\operatorname{Spec} \mathcal{R}_{\mathfrak{n}},A} \otimes_A P^{\otimes(n-1)}$. The sublemma now yields that all the maps

$$\underline{\mathbb{1}}_{\operatorname{Spec} \mathcal{R}_{\mathfrak{n}},A} \otimes_A P^{\otimes(n-1)} \cong \underline{\tilde{\mathcal{G}}}_1^{\max}/(\pi) \longrightarrow \underline{\tilde{\mathcal{G}}}_2^{\max}/(\pi) \longrightarrow \dots \longrightarrow \underline{\tilde{\mathcal{G}}}_n^{\max}/(\pi) \quad (59)$$

are nil-isomorphisms. In analogy with the notation $\widetilde{\mathfrak{M}}_{\mathfrak{n}}$, let $\widetilde{\underline{\mathcal{F}}_{\mathfrak{n}}^{(n-1)}}$ denote the pullback of the sheaf $\underline{\mathcal{F}}_{\mathfrak{n}}^{(n-1)}$ to $\widetilde{\mathfrak{M}}_{\mathfrak{n}}$. By definition, the sheaf $\tilde{\underline{\mathcal{G}}}_n$ is the restriction of $(\widetilde{\underline{\mathcal{F}}_{\mathfrak{n}}^{(n-1)}})^{A\text{-an}}$ to the infinitesimal neighborhood of a single cusp. Thus by Proposition 11.20 c) the sequence (59) gives rise to a nil-isomorphisms

$$\bigoplus_c \underline{\mathbb{1}}_{c,A} \otimes_A P_c^{\otimes(n-1)} \longrightarrow (i'_{\mathcal{K}})^* j'_{\mathcal{K}\#} \widetilde{\underline{\mathcal{F}}_{\mathfrak{n}}^{(n-1)}},$$

where the sum is over all components c of $\mathfrak{M}_{\mathcal{K}}^{\infty}$ and P_c depends uniquely on c (but not on n). By the remark following the statement of Theorem 11.19, the proof of Theorem 11.19 is finished in the case $\mathcal{K} = \mathcal{K}(\mathfrak{n})$ and $\mathfrak{n}$ principal.

Let now $\mathcal{K}$ be arbitrary admissible. Let $\mathfrak{n}$ be a conductor of $\mathcal{K}$ such that $\mathfrak{n}$ is principal and $V(\mathfrak{n})$ is minimal. In particular, $\mathcal{K}(\mathfrak{n}) \subset \mathcal{K}$. Then there is an associated Galois cover $\overline{\mathfrak{M}}_{\mathfrak{n}} \to \overline{\mathfrak{M}}_{\mathcal{K}}$ over $\operatorname{Spec} A(\mathfrak{n})$ whose Galois group we denote by G. In the notation of the previous paragraph, it follows that $\widetilde{\underline{\mathcal{F}}_{\mathcal{K}}^{(n-1)}}$ is the Galois descent under G of $\widetilde{\underline{\mathcal{F}}_{\mathfrak{n}}^{(n-1)}}$. Passing to the attached A-analytic τ-sheaves and considering maximal extensions, it follows easily that

$$\Big(\Big(\widetilde{\underline{\mathcal{F}}_{\mathcal{K}}^{(n-1)}}\Big)^{\max}\Big)^{A\text{-an}} \subset \Big(\Big(\Big(\widetilde{\underline{\mathcal{F}}_{\mathfrak{n}}^{(n-1)}}\Big)^{\max}\Big)^{A\text{-an}}\Big)^{G}.$$

Clearly $\underline{\mathbb{1}}_{\widetilde{\mathfrak{M}}_{\mathcal{K}},A} = \underline{\mathbb{1}}^{G}_{\widetilde{\mathfrak{M}}_{\mathfrak{n}},A}$. Also in any G-orbit of components c of $\mathfrak{M}_{\mathcal{K}}^{\infty}$, the module P_c is independent of the cusp c. This yields a for $? \in \{\mathcal{K}, \mathcal{K}(\mathfrak{n})\}$ a monomorphism

$$\bigoplus_{c_?} \underline{\mathbb{1}}_{c_?,A}^{A\text{-an}} \otimes_A P_{c_?} \longrightarrow \Big(\Big(\widetilde{\underline{\mathcal{F}}_{?}^{(n-1)}}\Big)^{\max}\Big)^{A\text{-an}},$$

where the sum is over the components $c_?$ of $\mathfrak{M}_?^{\infty}$. Let $\tilde{\underline{\mathcal{C}}}_?$ denote its cokernel.

Then $\tilde{\underline{\mathcal{C}}}_{\mathcal{K}} \subset \tilde{\underline{\mathcal{C}}}_{\mathfrak{n}}^{G} \subset \tilde{\underline{\mathcal{C}}}_{\mathfrak{n}}$ as finitely generated A-analytic τ-sheaves on $\widetilde{\mathfrak{M}}_{\mathcal{K}}$. By the result in the case $\mathcal{K} = \mathcal{K}(\mathfrak{n})$ there exists $m \in \mathbb{N}$ such that $\tau^m \tilde{\underline{\mathcal{C}}}_{\mathfrak{n}} \subset \pi \tilde{\underline{\mathcal{C}}}_{\mathfrak{n}}$. Thus by the Artin-Rees lemma, as in the proof of Sublemma 11.29, there exists an $m' \in \mathbb{N}$ such that $\tau^{m'} \tilde{\underline{\mathcal{C}}}_{\mathcal{K}} \subset \pi \tilde{\underline{\mathcal{C}}}_{\mathcal{K}}$. This shows that τ is nilpotent on $i'^*_{\mathcal{K}}(\tilde{\underline{\mathcal{C}}}_{\mathcal{K}})$. It follows as above that

$$\underline{\mathcal{F}}_{\mathcal{K}}^{(n-1),\infty} \longrightarrow i'^*_{\mathcal{K}}\Big(\Big(\widetilde{\underline{\mathcal{F}}_{\mathcal{K}}^{(n-1)}}\Big)^{\max}\Big)^{A\text{-an}} \overset{(\text{Prop.11.18})}{\cong} i'^*_{\mathcal{K}}\Big(\Big(\widetilde{\underline{\mathcal{F}}_{\mathcal{K}}^{(n-1)}}\Big)^{\max}\Big)$$

is a nil-isomorphism. ∎

While in general it cannot be expected that the formation of maximal extension commutes with pullback, we have the following special result on it commuting with base change in the situation we are interested in.

Theorem 11.30 *For* $? \in \{K, K_\infty\}$ *the* τ*-sheaf* $j_{\mathcal{K}\#}(\mathfrak{b}_?^* \underline{\mathcal{F}}_{\mathcal{K}}^{(n)})$ *is locally free of rank* n *and the following natural inclusions are a nil-isomorphisms*

$$\mathfrak{b}_?^*(i_{\mathcal{K}*} \underline{\mathcal{F}}_{\mathcal{K}}^{(n),\infty}) \longrightarrow \mathfrak{b}_?^*(j_{\mathcal{K}\#} \underline{\mathcal{F}}_{\mathcal{K}}^{(n)}) \longrightarrow j_{\mathcal{K}\#}(\mathfrak{b}_?^* \underline{\mathcal{F}}_{\mathcal{K}}^{(n)}).$$

Proof: The first assertion follows immediately from Theorem 11.15. Also the map $\mathfrak{b}_?^*(i_{\mathcal{K}*} \underline{\mathcal{F}}_{\mathcal{K}}^{(n),\infty}) \longrightarrow \mathfrak{b}_?^*(j_{\mathcal{K}\#} \underline{\mathcal{F}}_{\mathcal{K}}^{(n)})$ is clearly a nil-isomorphism, since this property is being preserved under pullbacks. Therefore it suffices to show that the morphism $\mathfrak{b}_?^*(i_{\mathcal{K}*} \underline{\mathcal{F}}_{\mathcal{K}}^{(n),\infty}) \longrightarrow j_{\mathcal{K}\#}(\mathfrak{b}_?^* \underline{\mathcal{F}}_{\mathcal{K}}^{(n)})$ is a nil-isomorphism. This assertion may be shown on infinitesimal neighborhoods of cusps, and so it suffices to consider the base change $\mathfrak{b}_? : \operatorname{Spec} \mathcal{R}_{\mathfrak{n}}[[\pi]] \to \operatorname{Spec}(? \otimes_A \mathcal{R}_{\mathfrak{n}})[[\pi]]$. Since the cases $? = K$ and $? = K_\infty$ have the same proof, for simplicity of notation, we give it in the case $? = K$.

Observe first that the pullback under $\mathfrak{b}_K$ of the sequence 55 yields the analogous sequence on $\operatorname{Spec}(K \otimes_A \mathcal{R}_{\mathfrak{n}})[[\pi]]$. Taking symmetric powers yields the exact sequence:

$$0 \longrightarrow \mathfrak{b}_K^* \underline{\tilde{\mathcal{H}}}^{\otimes n} \cong \mathbb{1}^{A\text{-an}}_{(K \otimes \mathcal{R}_{\mathfrak{n}})((\pi)),A} \otimes_A P^{\otimes n} \longrightarrow \operatorname{Sym}^n(\mathfrak{b}_K^* \underline{\tilde{\mathcal{M}}}_2) \longrightarrow \mathfrak{b}_K^* \underline{\mathcal{C}}_n \longrightarrow 0.$$

This shows that

$$(\mathfrak{b}_K^* \underline{\tilde{\mathcal{H}}})^{\otimes n, \max} \cong \mathbb{1}^{A\text{-an}}_{(K \otimes \mathcal{R}_{\mathfrak{n}})[[\pi]],A} \otimes_A P^{\otimes n} \hookrightarrow \operatorname{Sym}^n(\mathfrak{b}_K^* \underline{\tilde{\mathcal{M}}}_2)^{\max}$$

is a monomorphism. Let $\underline{\mathcal{C}}'_n$ be the cokernel of the latter map. Using the maximality of $(\mathfrak{b}_K^* \underline{\tilde{\mathcal{H}}})^{\max}$, one shows as before that $\underline{\mathcal{C}}'_n$ is torsion free and that the right exact sequence

$$(\mathfrak{b}_K^* \underline{\tilde{\mathcal{H}}})^{\otimes n, \max}/(\pi) \longrightarrow \operatorname{Sym}^n(\mathfrak{b}_K^* \underline{\tilde{\mathcal{M}}}_2)^{\max}/(\pi) \longrightarrow \underline{\mathcal{C}}'_n/(\pi) \longrightarrow 0$$

is also left exact.

Let us assume that $\underline{\mathcal{C}}'_n/(\pi)$ is not nilpotent. Because $\underline{\mathcal{C}}'_n \to \underline{\mathcal{C}}_n^{\max}$ is injective and τ is nilpotent on the special fiber of any sub τ-sheaf of $\pi(\underline{\mathcal{C}}_n^{\max})$, it follows that the image of $\underline{\mathcal{C}}'_n/(\pi)$ in the locally free sheaf $\underline{\mathcal{C}}_n^{\max}/(\pi)$ is non-trivial. This implies that $\operatorname{Sym}^n(\mathfrak{b}_K^* \underline{\tilde{\mathcal{M}}}_2)^{\max}/(\pi)$ contains a τ-subsheaf of rank at least two on which τ is not injective.

Following the discussion after [13], Rem. 1.24, one can prove the following extension of Propositon 11.20(b):

Lemma 11.31 *Let $\underline{\tilde{\mathcal{F}}}$ be an A-analytic torsion free τ-sheaf on* $\operatorname{Spec} R((\pi))$. *Suppose $\underline{\tilde{\mathcal{G}}}$ is a maximal locally free extension of $\underline{\tilde{\mathcal{F}}}$. Then there exist*

(i) a finite set of points S of $\operatorname{Spec} R \otimes A$ *and a finite number of discs D_i on the rim of the rigid space $\mathfrak{D}_A$,*

(ii) a locally free analytic τ-subsheaf $\underline{\tilde{\mathcal{G}}}'$ of $\underline{\tilde{\mathcal{G}}}$ on the formal scheme $X :=$ $\operatorname{Spm}(R[[\pi]] \otimes A)^{A\text{-an}} \smallsetminus S \smallsetminus \bigcup_i D_i$

such that $\underline{\tilde{\mathcal{G}}}'$ has good reduction and the induced map $i_R^* \underline{\tilde{\mathcal{G}}}' \hookrightarrow i_R^* \underline{\tilde{\mathcal{G}}}$ is an injective nil-isomorphism on $\operatorname{Spec}(R \otimes A) \smallsetminus S$.

In particular, for infinitely many primes l of A, the prime $R[[\pi]]] \otimes l$ is contained in X.

By the lemma, on a formal scheme X as above there exists a locally free τ-subsheaf $\underline{\tilde{\mathcal{G}}}'$ of $\operatorname{Sym}^n(\mathfrak{b}_K^* \underline{\tilde{\mathcal{M}}}_2)^{\max}$ of rank at least two. We complete this at any prime l of A that lies not in X. Via the duality between the l-adic Tate-modules of τ-sheaves and forml l-adic τ-sheaves, this implies that the submodule of inertia invariants of $\operatorname{Ta}_l(\operatorname{Sym}^n(\mathfrak{b}_K^* \underline{\tilde{\mathcal{M}}}_2))$ is of rank at least two. However this Tate-module is the n-th symmetric power of the Tate-module of a Drinfeld-module on $K \otimes \mathcal{R}_{\mathfrak{n}}((\pi))$ which has stable reduction of rank 1 modulo π. For almost all l, the inertia invariants of these symmetric powers are of dimension one over A_l, which yields a contradiction. ∎

11.3 Maximal extensions for formal and finite coefficients

In the previous subsections, we considered maximal extensions for τ-sheaves over A. However for the comparison with the étale site, it is necessary to work with finite and formal coefficients. Therefore in this subsection, we extend the results of Subsection 11.1 to such coefficients and discuss the transition from crystals to étale sheaves for maximal extensions.

We fix a proper non-zero ideal $\mathfrak{n}$ and a maximal ideal $\mathfrak{p}$ of A. For any scheme X over k we define $X\hat{\otimes}A_{\mathfrak{p}}$ as the formal scheme obtained as the completion of $X \times \operatorname{Spec} A$ along the ideal sheaf $\mathcal{O}_X \otimes \mathfrak{p}$, see [53]. Let $\sigma \times \mathrm{id}$ denote the morphism on $X\hat{\otimes}A_{\mathfrak{p}}$ induced from the inverse limit system $\sigma_X \times \mathrm{id}_{A/\mathfrak{p}^n}$.

Definition 11.32 *A* formal τ-sheaf $\underline{\tilde{\mathcal{F}}} = (\tilde{\mathcal{F}}, \tau)$ on X over $A_{\mathfrak{p}}$ *consists of a coherent sheaf $\tilde{\mathcal{F}}$ on the formal scheme $X\hat{\otimes}A_{\mathfrak{p}}$ together with an $\mathcal{O}_X\hat{\otimes}A_{\mathfrak{p}}$-linear morphism*

$$\tau\colon (\sigma \times \mathrm{id})^*\tilde{\mathcal{F}} \longrightarrow \tilde{\mathcal{F}}.$$

The category of formal τ-sheaves on X over $A_{\mathfrak{p}}$ is denoted by $\mathbf{Coh}^{\mathbf{f}}_{\tau}(X, A_{\mathfrak{p}})$.

We omit any definition of crystals in the formal context, as it will not be needed. Let us fix a dense open immersion $j : U \to X$ and a closed complement $i : Z \to X$. Also, let $\underline{\mathcal{F}}$ and $\underline{\tilde{\mathcal{F}}}$ be τ-sheaves in $\mathbf{Coh}_{\tau}(U, A/\mathfrak{n})$ and $\mathbf{Coh}^{\mathbf{f}}_{\tau}(U, A_{\mathfrak{p}})$, respectively.

Definition 11.33 *Any $\underline{\mathcal{G}} \in \mathbf{Coh}_{\tau}(X, A/\mathfrak{n})$ with $j^*\underline{\mathcal{G}} \cong \underline{\mathcal{F}}$ is called an* extension *of $\underline{\mathcal{F}}$.*

Define $j_{\#}\underline{\mathcal{F}} \subset j_*\underline{\mathcal{F}} \in \mathbf{QCoh}_{\tau}(X, A/\mathfrak{n})$ as the union of all extensions $\underline{\mathcal{G}}$ of $\underline{\mathcal{F}}$, or equivalently as the union of all coherent τ-subsheaves of $j_*\underline{\mathcal{G}}$.

If $j_{\#}\underline{\mathcal{F}}$ is coherent, it is called the maximal extension of $\underline{\mathcal{F}}$ with respect to j.

Analogously one defines $j_{\#}\underline{\tilde{\mathcal{F}}}$ and the notion of an extension and of a maximal extension for formal τ-sheaves.

If a maximal extension exists, it is unique up to unique isomorphism. For τ-sheaves over $A/\mathfrak{n}$ on U, there always exists an extension to X.

As in Subsection 11.1, one can show that the existence of a maximal extension is a local condition on X:

Proposition 11.34 *Let $\{U_i\}$ be an open cover of X. Suppose $\underline{\mathcal{G}}$ is an extension of $\underline{\mathcal{F}}$ such that each $\underline{\mathcal{G}}_{|U_i}$ is a maximal extension of $\underline{\mathcal{F}}_{|U_i \cap U}$. Then $\underline{\mathcal{G}}$ is a maximal extension of $\underline{\mathcal{F}}$.*

The analogous assertion holds in the category of formal τ-sheaves over $A_{\mathfrak{p}}$.

Without proof, we state the following analogue of Corollary 11.14.

Proposition 11.35 *Let $\underline{\mathcal{F}} \in \mathbf{Coh}_{\tau}(U, A/\mathfrak{n})$ be locally free and assume that X is normal. Let $z_1, \ldots, z_r$ denote those generic points of Z which are of height 1 in X. For each $l \in \{1, \ldots, r\}$ consider the pullback diagram*

$$\begin{tikzcd}
U \arrow[r, "j"] & X \\
\operatorname{Spec} \mathcal{O}_{X,z_l} \smallsetminus \{z_l\} \arrow[u, "f_l'"] \arrow[r, "j_l"'] & \operatorname{Spec} \mathcal{O}_{X,z_l}. \arrow[u, "f_l"']
\end{tikzcd}$$

Let $\underline{\mathcal{F}}_l$ denote the pullback of $\underline{\mathcal{F}}$ to $\operatorname{Spec} \mathcal{O}_{X,z_l} \smallsetminus \{z_l\}$. If each of the $\underline{\mathcal{F}}_l$ has a maximal extension to $\operatorname{Spec} \mathcal{O}_{X,z_l}$, then $\underline{\mathcal{F}}$ has a maximal extension to X.

The analogous assertion holds in the category of formal τ-sheaves over $A_{\mathfrak{p}}$.

Note that in the proof for τ-sheaves over $A/\mathfrak{n}$ one has to regard the underlying sheaf as a coherent sheaf over the normal scheme X, since $X \times \operatorname{Spec} A/\mathfrak{n}$ will in general not be normal.

Theorem 11.36 *Suppose X is a normal scheme, $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U, A/\mathfrak{n})$ and the morphism $\tau_{\mathcal{F}}$ is injective. Then $j_\#\underline{\mathcal{F}}$ is coherent, i.e., $\underline{\mathcal{F}}$ has a maximal extension to X.*

Proof: By the previous proposition, one may assume that $X = \operatorname{Spec} \mathbb{V}$ for a discrete valuation ring $\mathbb{V}$ and $U \to X$ is the map of schemes corresponding to the generalization map $\mathbb{V} \to \mathbb{K}$ where $\mathbb{K}$ is the fraction field of $\mathbb{V}$. Suppose we have $\underline{\mathcal{M}}' \subset \underline{\mathcal{M}} \subset j_\#\underline{\mathcal{F}}$. Because τ is injective on $\underline{\mathcal{F}}$, it is injective on $j_*\underline{\mathcal{F}}$ and hence on $\underline{\mathcal{M}}'$ and $\underline{\mathcal{M}}$. Let $\underline{\mathcal{M}}'' := \underline{\mathcal{M}}/\underline{\mathcal{M}}'$ and consider the following commutative diagram with exact rows

$$\begin{tikzcd}
 & \sigma^*\underline{\mathcal{M}}' \arrow[r] \arrow[d, hook, "\tau'_{\mathcal{M}}"] & \sigma^*\underline{\mathcal{M}} \arrow[r] \arrow[d, hook, "\tau_{\mathcal{M}}"] & \sigma^*\underline{\mathcal{M}}'' \arrow[r] \arrow[d, "\tau''_{\mathcal{M}}"] & 0 \\
0 \arrow[r] & \underline{\mathcal{M}}' \arrow[r] & \underline{\mathcal{M}} \arrow[r] & \underline{\mathcal{M}}'' \arrow[r] & 0.
\end{tikzcd}$$

Clearly $\underline{\mathcal{M}}''$ is of finite length and so are the cokernels of $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{M}'}$. Using the Snake Lemma, one finds

$$\operatorname{len} \operatorname{Coker}(\tau_{\mathcal{M}'}) - \operatorname{len} \operatorname{Coker}(\tau_{\mathcal{M}}) = \operatorname{len} \sigma^*\underline{\mathcal{M}}'' - \operatorname{len} \underline{\mathcal{M}}'' = (q-1) \operatorname{len} \underline{\mathcal{M}}'' > 0.$$

It is easy to see that $\underline{\mathcal{F}}$ has an extension, and that the union of any two extensions is again an extension. Hence by the above inequality, any extension $\underline{\mathcal{M}}'$ for which $\operatorname{len} \operatorname{Coker} \tau_{\mathcal{M}'}$ is minimal will be a (the) maximal extension of $\underline{\mathcal{F}}$. ∎

Suppose $\tilde{\underline{\mathcal{F}}} \in \mathbf{Coh}^{\mathbf{f}}_\tau(U, A_\mathfrak{p})$. We cannot prove any interesting results for the natural transformation $(j_\#\tilde{\underline{\mathcal{F}}}) \otimes_A A/\mathfrak{n} \to j_\#(\tilde{\underline{\mathcal{F}}} \otimes_A A/\mathfrak{n})$. However as the following result shows, passing to the formal site greatly improves the situation.

Theorem 11.37 *Suppose X is a normal scheme and $\tilde{\underline{\mathcal{F}}}$ a locally free formal τ-sheaf on U over $A_\mathfrak{p}$ such that the morphism $(\sigma_U \times \mathrm{id})^*\tilde{\mathcal{F}} \xrightarrow{\tau_{\tilde{\mathcal{F}}}} \tilde{\mathcal{F}}$ is injective. Then the following hold:*

(i) $j_\#\tilde{\underline{\mathcal{F}}}$ is coherent.

(ii) If the cokernel of $\tau_{\tilde{\underline{\mathcal{F}}}}$ is $A_\mathfrak{p}$-torsion free, then

$$j_\#\tilde{\underline{\mathcal{F}}} \cong \varprojlim j_\#(\tilde{\underline{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^n)).$$

(iii) If $\tilde{\underline{\mathcal{F}}} = \underline{\mathcal{F}}'\hat{\otimes}_A A_\mathfrak{p}$ for some τ-sheaf $\underline{\mathcal{F}}' \in \mathbf{Coh}_\tau(U, A)$ such that $\tau_{\mathcal{F}'}$ is injective and $\operatorname{Coker} \tau_{\mathcal{F}'}$ is A-torsion free, then $(j_\#\underline{\mathcal{F}}')\hat{\otimes}_A A_\mathfrak{p} \hookrightarrow j_\#\tilde{\underline{\mathcal{F}}}$ is a monomorphism and a nil-isomorphism in codimensions zero and one (on the scheme X).

(iv) Suppose in the situation of part (iii) that Z is normal and that there is a τ-subsheaf $\underline{\mathcal{G}}'$ of $i^*j_\#\underline{\mathcal{F}}'$ which has good reduction at all points of Z and such that $\underline{\mathcal{G}} \hookrightarrow i^*j_\#\underline{\mathcal{F}}'$ is a nil-isomorphism. Then the monomorphism $(j_\#\underline{\mathcal{F}}')\hat{\otimes}_A A_\mathfrak{p} \hookrightarrow j_\#\tilde{\underline{\mathcal{F}}}$ is a nil-isomorphism.

Proof: (i): For the coherence of $j_\#\tilde{\underline{\mathcal{F}}}$, it suffices by Proposition 11.35, to consider the case where $U = \operatorname{Spec} \mathbb{V}$ such that $\mathbb{V}$ is a discrete valuation ring. In this situation, coherence is proved in exactly the same way as in Proposition 11.17, where this time one applies Lemma 11.16 with $R = \mathbb{V}\hat{\otimes} A_\mathfrak{p}$.

(ii): Let us now assume that $\operatorname{Coker} \tau_{\tilde{\underline{\mathcal{F}}}}$ is $A_\mathfrak{p}$-torsion free. This assumption implies that τ is injective on $\tilde{\underline{\mathcal{F}}} \otimes_A A/\mathfrak{p}^n$ for any n. Thus by Theorem 11.36, the

τ-sheaf $j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/\mathfrak{p}^n)$ is coherent. In an obvious way, the $j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^n))$ form an inverse system of τ-sheaves. Moreover there is an inclusion

$$j_\#\underline{\tilde{\mathcal{F}}} \otimes_{A_\mathfrak{p}} A/\mathfrak{p}^n \hookrightarrow j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^n))$$

whose image lies in the image of $j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^{n'})) \otimes_A /\mathfrak{p}^n \to j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^n))$ for all $n' \geq n$. The τ-sheaves $j_\#(\underline{\tilde{\mathcal{F}}} \otimes A/\mathfrak{p}^n)$ are determined by their stalks at the points of Z of codimension one in X. By considering these points, the above observations imply that the inverse system $(j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^n)))_n$ satisfies the Mittag-Leffler condition. Therefore in

$$j_\#\underline{\tilde{\mathcal{F}}} \longrightarrow \varprojlim j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^n)) \longrightarrow j_*\underline{\tilde{\mathcal{F}}},$$

both maps to the right hand τ-sheaf are injective. As $j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}))$ is coherent, so is the inverse limit in the middle, and the definition of $j_\#\underline{\tilde{\mathcal{F}}}$ shows that the left hand morphism is an isomorphism.

(iv): For this we assume part (iii) to be proved. By our assumptions and Lemma 11.16, we have the following monomorphisms of torsion free τ-sheaves:

$$\underline{\mathcal{G}}'\hat{\otimes}_A A_\mathfrak{p} \longleftarrow\!\!\!\!\hookrightarrow (i^* j_\# \underline{\mathcal{F}}')\hat{\otimes}_A A_\mathfrak{p} \hookrightarrow i^* j_\# \underline{\tilde{\mathcal{F}}}. \tag{60}$$

The left hand morphism is a nil-isomorphism, and by part (iii), the right hand morphism is a nil-isomorphsm on the generic points of Z; say they are $\eta_1, \ldots, \eta_l$. Because $\underline{\mathcal{G}}'$ has everywhere good reduction and Z is normal, $\underline{\mathcal{G}}'$ is the maximal extension of $\underline{\mathcal{G}}'_{|V}$ for any dense open $V \subset Z$, cf. Proposition 11.9. In particular this implies that

$$\underline{\mathcal{G}}'\hat{\otimes}_A A_\mathfrak{p} = i^* j_\# \underline{\tilde{\mathcal{F}}} \cap \bigoplus_j (\underline{\mathcal{G}}'\hat{\otimes}_A A_\mathfrak{p})_{\eta_j}.$$

Therefore the cokernel of $\alpha\colon \underline{\mathcal{G}}'\hat{\otimes}_A A_\mathfrak{p} \hookrightarrow i^* j_\# \underline{\tilde{\mathcal{F}}}$ is a torsion free τ-sheaf. As this cokernel is generically nilpotent, it is nilpotent and thus α is a nil-isomorphism. This easily implies the same for the morphisms in (60).

(iii): By Proposition 11.35, we may assume that X is the spectrum of a discrete valuation ring $\mathbb{V}$ with fraction field $\mathbb{K}$, residue field $\tilde{k}$ and a uniformizer π, and that $j\colon U \hookrightarrow X$ is given by the generalization map $\mathbb{V} \to \mathbb{K}$. We need to prove a strengthening of [13], Thm. 4.7.

To work in the notation of loc. cit., define B as the localization of $\mathbb{V}\hat{\otimes}A_\mathfrak{p}$ at the set $\mathbb{V}\hat{\otimes}A_\mathfrak{p} \smallsetminus \bigcup \mathfrak{p}_i$, where the $\mathfrak{p}_i$ are the primes associated to (π). Let Q be the quotient ring of B. By $\hat{B}$ we denote the completion of B at the ideal πB and we set $\hat{Q} := \hat{B} \otimes_B Q$. Furthermore, λ will denote the quotient ring of $k\hat{\otimes}A_\mathfrak{p}$, i.e., $\lambda \cong B/\pi B$. We depict the relevant rings in the following commutative diagram

$$\begin{tikzcd}
k\hat{\otimes}A_\mathfrak{p} \arrow[d, hook] & \mathbb{V}\hat{\otimes}A_\mathfrak{p} \arrow[l, two heads] \arrow[r, hook] \arrow[d, hook] & (\mathbb{V}\hat{\otimes}A_\mathfrak{p})\left[\frac{1}{\pi}\right] \arrow[d, hook] \\
\lambda \arrow[d, equal] & B \arrow[l, two heads] \arrow[r, hook] \arrow[d, hook] & Q \arrow[d, hook] \\
\lambda & \hat{B} \arrow[l, two heads] \arrow[r, hook] & \hat{Q}.
\end{tikzcd} \tag{61}$$

Note that $\mathbb{V}\hat{\otimes}A_\mathfrak{p}$ and $(\mathbb{V}\hat{\otimes}A_\mathfrak{p})[1/\pi]$ take the role of $R[[t]]$ and $K \otimes_R R[[t]]$ of loc. cit., but that the former rings may be finite products of rings of the form $R[[t]]$ or $K \otimes_R R[[t]]$, respectively.

Define $N' \subset N$ as the global sections of $(j_\#\underline{\mathcal{F}}')\hat{\otimes}_A A_\mathfrak{p} \subset j_\#\underline{\tilde{\mathcal{F}}}$ and $\bar{N}' := N' \otimes_\mathbb{V} \tilde{k}$ and $\bar{N} := N \otimes_\mathbb{V} \tilde{k}$ as their reductions modulo π. The latter are τ-modules on k over $A_\mathfrak{p}$ and there is a natural map $\bar{N}' \to \bar{N}$, which is not necessarily an inclusion. By [13], Rem. 1.7, there are compatible short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \bar{M}' \arrow[r] \arrow[d, hook] & \bar{N}' \arrow[r] \arrow[d] & \bar{N}'_{\mathrm{nil}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{M} \arrow[r] & \bar{N} \arrow[r] & \bar{N}_{\mathrm{nil}} \arrow[r] & 0
\end{tikzcd}$$

of $k\hat{\otimes}A_{\mathfrak{p}}$-modules, where τ is nilpotent on the right hand terms and injective on the left hand terms. Note that by [13], Lem. 1.10, the left vertical map is injective. We claim that it is an isomorphism. Observe that this claim is equivalent to the assertion that $i^*((j_\#\underline{\mathcal{F}}')\hat{\otimes}_A A_{\mathfrak{p}}) \subset i^*(j_\#\underline{\tilde{\mathcal{F}}})$ is a nil-isomorphism, which in turn implies that $(j_\#\underline{\mathcal{F}}')\hat{\otimes}_A A_{\mathfrak{p}} \subset j_\#\underline{\tilde{\mathcal{F}}}$ is a nil-isomorphism, as asserted.

To prove the claim, we first note that by [13], Thm. 4.7, one has

$$\bar{M}' \otimes_{k\hat{\otimes}A_{\mathfrak{p}}} \lambda \cong \bar{M} \otimes_{k\hat{\otimes}A_{\mathfrak{p}}} \lambda =: \bar{M}_\lambda.$$

By slightly altering the proof of [13], Prop. 1.22, one can show that there exist unique submodules $\hat{M}'$ of $\hat{N}' := N' \otimes_{\mathbb{V}\hat{\otimes}A_{\mathfrak{p}}} \hat{B}$ and $\hat{M}$ of $\hat{N} := N \otimes_{\mathbb{V}\hat{\otimes}A_{\mathfrak{p}}} \hat{B}$ which reduce modulo π to $\bar{M}_\lambda$. Also note that $\hat{M}$ and $\hat{M}'$ are direct summands of $\hat{N}$ and $\hat{N}'$, respectively.

Because $\hat{M}' \subset \hat{M}$ and they have the same reduction mod π, on which τ is an isomorphism, they agree inside $\hat{N}$, cf. [13], Lem. 1.10. Now Lemma 11.16 shows that $N = N \otimes_{\mathbb{V}\hat{\otimes}A_{\mathfrak{p}}} B \cap N[1/\pi]$ and analogously for N'. By considering the bottom right square of (61), one can furthermore show that inside $N \otimes \hat{Q}$ one has $N = N \otimes_{\mathbb{V}\hat{\otimes}A_{\mathfrak{p}}} \hat{B} \cap N[1/\pi]$ and the analogous assertion for N'. Since N and N' agree when restricted to $\mathbb{K}$, it follows that $N[1/\pi] = N'[1/\pi]$, and hence that

$$M' := \hat{M}' \cap N'[1/\pi] = M := \hat{M} \cap N[1/\pi] \subset N' \subset N.$$

It is easy to see that $\bar{M}'$ is the reduction of M' modulo π and $\bar{M}$ that of M. Thus it follows $\bar{M}' = \bar{M}$, as claimed. ∎

In the notation of [4], Ch. 8, the functor $j_\#$ on $\mathbf{Coh}_\tau(X, A/\mathfrak{n})$ is denoted $\mathrm{Ind} \circ j_*$ and the following is shown in loc. cit. as part of the comparison between crystals and étale sheaves for finite coefficients.

Proposition 11.38 *There is an isomorphism of functors*

$$(j_\#\underline{\mathcal{F}})_{\text{ét}} \cong j_{*\text{ét}}(\underline{\mathcal{F}}_{\text{ét}}) \colon \mathbf{Coh}_\tau(U, A/\mathfrak{n}) \to \mathbf{\acute{E}t}(X, A/\mathfrak{n}),$$

which maps nil-isomorphisms to isomorphisms. If $j_\#\underline{\mathcal{F}}$ is coherent, then the étale sheaf $j_{*\text{ét}}(\underline{\mathcal{F}}_{\text{ét}})$ is constructible.

By an inverse limit argument, the above proposition can be transferred to formal sites. For this we note that the functor $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}$ extends in a natural way to a functor $\underline{\tilde{\mathcal{F}}} \mapsto \underline{\tilde{\mathcal{F}}}_{\text{ét},\mathfrak{p}}$ from coherent formal τ-sheaves $\underline{\tilde{\mathcal{F}}}$ over $A_{\mathfrak{p}}$ to constructible étale $A_{\mathfrak{p}}$-sheaves

$$\underline{\tilde{\mathcal{F}}}_{\text{ét},\mathfrak{p}} := \varprojlim_n (\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^n))_{\text{ét}}.$$

Theorem 11.39 *Suppose X is normal and $\underline{\tilde{\mathcal{F}}} \in \mathbf{Coh}^{\mathbf{f}}_\tau(X, A_{\mathfrak{p}})$ is locally free such that $\tau_{\tilde{\mathcal{F}}}$ is injective and* $\mathrm{Coker}(\tau_{\tilde{\mathcal{F}}})$ *is torsion free over $A_{\mathfrak{p}}$. Then there are functorial isomorphisms*

$$j_{*\text{ét}}(\underline{\tilde{\mathcal{F}}}_{\text{ét},\mathfrak{p}}) \cong \Big(\varprojlim_n j_\#(\underline{\tilde{\mathcal{F}}} \otimes_A A/(\mathfrak{p}^n))\Big)_{\text{ét}} \cong \Big(j_\#\underline{\tilde{\mathcal{F}}}\Big)_{\text{ét},\mathfrak{p}}.$$

Proof: The proof follows from the above proposition and Theorem 11.37. ∎

As a consequence of Theorems 11.39 and 11.37, we obtain:

Corollary 11.40 *Suppose X is normal and $\underline{\mathcal{G}} \in \mathbf{Coh}_\tau(U,A)$ is such that $\tau_{\mathcal{G}}$ is injective and has cokernel which is A-torsion free. Then there is a monomorphism*

$$(j_\# \underline{\mathcal{F}} \hat{\otimes} A_{\mathfrak{p}})_{\text{ét},\mathfrak{p}} \hookrightarrow j_{*\text{ét}}(\underline{\mathcal{F}} \hat{\otimes} A_{\mathfrak{p}})_{\text{ét},\mathfrak{p}} \cong (j_\#(\underline{\mathcal{F}} \hat{\otimes} A_{\mathfrak{p}}))_{\text{ét},\mathfrak{p}},$$

whose cokernel is supported in codimension at least 2. If furthermore $i^* j_\# \underline{\mathcal{F}}$ contains a nil-isomorphic τ-sheaf with everywhere good reduction, then the above map is an isomorphism.

Remark 11.41 If $\mathfrak{p} \in \operatorname{Spec} A$ corresponds to the place v of K, then we also write $\underline{\tilde{\mathcal{F}}}_{\text{ét},v}$ for $\underline{\tilde{\mathcal{F}}}_{\text{ét},\mathfrak{p}}$.

11.4 Maximal extensions of rigid τ-sheaves and crystals

In this subsection, we will consider maximal extensions over an analytic base. As in Section 8, all schemes $\mathfrak{X}, \mathfrak{U}, \mathfrak{Y}, \ldots$, are over an extension $L \subset \mathbb{C}_\infty$ of K_∞. Let us fix throughout a Zariski-open immersion $j : \mathfrak{U} \to \mathfrak{X}$ of rigid spaces. We also choose a closed complement $i : \mathfrak{Z} \to \bar{\mathfrak{U}}$ of $\mathfrak{U}$ in the Zariski closure $\bar{\mathfrak{U}} \subset \mathfrak{X}$ of $\mathfrak{U}$. Let $? \in \{A, \mathfrak{D}_A, \mathfrak{A}\}$.

Definition 11.42 *Any $\underline{\tilde{\mathcal{G}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, ?)$ with $j^* \underline{\tilde{\mathcal{G}}} \cong \underline{\tilde{\mathcal{F}}}$ is called an* extension *of $\underline{\tilde{\mathcal{F}}}$.*

We define $j_\# \underline{\tilde{\mathcal{F}}} \subset j_* \underline{\tilde{\mathcal{F}}}$ as the union of all extensions $\underline{\tilde{\mathcal{G}}}$ of $\underline{\tilde{\mathcal{F}}}$.

If $j_\# \underline{\tilde{\mathcal{F}}}$ is coherent, it is called the maximal extension of $\underline{\tilde{\mathcal{F}}}$ with respect to j.

By modifying the proof of Proposition 11.3, one easily obtains the following characterization of a maximal extension.

Proposition 11.43 *A coherent rigid τ-sheaf $\underline{\tilde{\mathcal{G}}}$ on $\mathfrak{X}$ is a maximal extension of $\underline{\tilde{\mathcal{F}}}$ if and only if for all $\underline{\tilde{\mathcal{H}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, A)$, the canonical map*

$$\operatorname{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\mathfrak{X},?)}(\underline{\tilde{\mathcal{H}}}, \underline{\tilde{\mathcal{G}}}) \longrightarrow \operatorname{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\mathfrak{U},?)}(j^* \underline{\tilde{\mathcal{H}}}, \underline{\tilde{\mathcal{F}}})$$

is an isomorphism.

Definition 11.44 *A crystal $\underline{\tilde{\mathcal{G}}} \in \widetilde{\mathbf{Crys}}(\mathfrak{X}, ?)$ is called an extension of $\underline{\tilde{\mathcal{F}}}$ if $j^* \underline{\tilde{\mathcal{G}}} \cong \underline{\tilde{\mathcal{F}}}$. It is called a* maximal extension *if in addition for all $\underline{\tilde{\mathcal{H}}} \in \mathbf{Crys}(\mathfrak{X}, ?)$, the canonical map*

$$\operatorname{Hom}_{\mathbf{Crys}(\mathfrak{X},?)}(\underline{\tilde{\mathcal{H}}}, \underline{\tilde{\mathcal{G}}}) \longrightarrow \operatorname{Hom}_{\mathbf{Crys}(\mathfrak{U},?)}(j^* \underline{\tilde{\mathcal{H}}}, \underline{\tilde{\mathcal{F}}})$$

is an isomorphism.

As in the algebraic case, the functor $j_\#$ is neither exact, nor preserves coherence of rigid τ-sheaves.

Because coherence is a local property, the following result is immediate:

Proposition 11.45 *Let $\{\mathfrak{U}_i\}$ be an admissible open cover of $\mathfrak{X}$. Suppose $\underline{\tilde{\mathcal{G}}}$ is an extension of $\underline{\tilde{\mathcal{F}}}$ such that each $\underline{\tilde{\mathcal{G}}}_{|\mathfrak{U}_i}$ is a maximal extension of $\underline{\tilde{\mathcal{F}}}_{|\mathfrak{U}_i \cap \mathfrak{U}}$. Then $\underline{\tilde{\mathcal{G}}}$ is a maximal extension of $\underline{\tilde{\mathcal{F}}}$.*

Using Proposition 8.4, the above proposition and the fact that any Zariski-open subset of an affinoid is an affinoid, one may carry over the proof of Proposition 11.5 to the rigid site. This yields:

Proposition 11.46 *Suppose $\underline{\tilde{\mathcal{F}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{U}, ?)$ has a maximal extension $\underline{\tilde{\mathcal{G}}} \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{X}, ?)$. Then the crystal represented by $\underline{\tilde{\mathcal{G}}}$ is a maximal extension of the crystal represented by $\underline{\tilde{\mathcal{F}}}$.*

Definition 11.47 *A rigid τ-sheaf $\underline{\tilde{\mathcal{G}}}$ on $\mathfrak{X}$ is said to have* good reduction on Z *if the map*

$$\tau_{i^*\tilde{\mathcal{G}}} \colon (\sigma \times \mathrm{id})^* i^* \tilde{\mathcal{G}} \to i^* \tilde{\mathcal{G}}$$

is injective.

A locally free extension $\underline{\tilde{\mathcal{G}}}$ of $\underline{\tilde{\mathcal{F}}} \in \mathbf{Coh}_\tau(\mathfrak{U}, ?)$ is called good *if $\tau_{\tilde{\mathcal{F}}}$ is injective and $\underline{\tilde{\mathcal{G}}}$ has good reduction to $\mathfrak{Y}$.*

Proposition 11.48 *If $\mathfrak{X}$ is a smooth curve and $\mathfrak{X} \smallsetminus \mathfrak{U}$ is finite, then any good extension is maximal.*

The proof is analogous to that of Proposition 11.9. It will be omitted, because the crucial part of the argument is similar to the one given in the proof of Proposition 11.50. In the rigid site all points are closed. Therefore it is not obvious how to carry over the proof of Proposition 11.9 to general normal domains $\mathfrak{X}$.

As a consequence, if $\underline{\tilde{\mathcal{G}}}$ is a rigid τ-sheaf attached to a family of A-motives on $\mathfrak{X}$ and $\mathfrak{U}$ is a dense open subset of $\mathfrak{X}$, then $\underline{\tilde{\mathcal{G}}}$ is the maximal extension of the restriction $j^*\underline{\tilde{\mathcal{G}}}$. Also, we note the following corollary which is analogous to Corollary 11.10.

Corollary 11.49 *Suppose $\mathfrak{X}$ is a smooth curve and $\mathfrak{X} \smallsetminus \mathfrak{U}$ is finite. Then the rigid τ-sheaf $\underline{\tilde{\mathbb{1}}}_{\mathfrak{X},?}$ is the maximal extension of $\underline{\tilde{\mathbb{1}}}_{\mathfrak{U},?}$.*

For curves, the following result gives a criterion for the functor $j_\#$ to commute with rigidification.

Proposition 11.50 *Suppose X is a smooth curve and $\underline{\mathcal{F}}$ a torsion free τ-sheaf on the dense open subset U of X. Suppose $j_\#\underline{\mathcal{F}}$ is coherent. Then the natural inclusion $(j_\#\underline{\mathcal{F}})^{\mathfrak{D}_A\text{-rig}} \hookrightarrow j_\#(\underline{\mathcal{F}}^{\mathfrak{D}_A\text{-rig}})$ on $\mathfrak{X} := X^{\mathrm{rig}}$ is an isomorphism. In particular, $j_\#(\underline{\mathcal{F}}^{\mathfrak{D}_A\text{-rig}})$ exists and $(i^*(j_\#\underline{\mathcal{F}}))^{\mathfrak{D}_A\text{-rig}} \cong i^* j_\#(\underline{\mathcal{F}}^{\mathfrak{D}_A\text{-rig}})$.*

Proof: We argue by contradiction and assume that there exists a rigid τ-sheaf $\underline{\tilde{\mathcal{G}}}$ on $\mathfrak{X}$ such that the inclusion $(j_\#\underline{\mathcal{F}})^{\mathfrak{D}_A\text{-rig}} \subset \underline{\tilde{\mathcal{G}}}$ is strict. Let x be a point in $\mathfrak{X} \smallsetminus \mathfrak{U}$ at which one has strict containment. Let $\mathbb{V}$ be the discrete valuation ring of the completion of $\mathcal{O}_{\mathfrak{X},x}$. By our choice of x and faithful flatness of $\mathcal{O}_{\mathfrak{X},x} \to \mathbb{V}$, the base change morphism $\beta \colon \operatorname{Spec} \mathbb{V} \to \mathfrak{X}$ preserves the strict containment, so that one has a strict inclusion

$$(\beta^*(j_\#\underline{\mathcal{F}}))^{A\text{-an}} \subset \beta^*\underline{\tilde{\mathcal{G}}}. \tag{62}$$

Because $\beta^*(j_\#\underline{\mathcal{F}})$ is a maximal extension of $\beta^*\underline{\mathcal{F}}$, Proposition 11.20(c) implies that $(\beta^*(j_\#\underline{\mathcal{F}}))^{A\text{-an}} = j_\#((\beta^*\underline{\mathcal{F}})^{A\text{-an}})$. Hence the coherent sheaf $\beta^*\underline{\mathcal{G}}$ must be contained in it, a contradiction to (62) being a strict containment. ∎

11.5 The maximal extension of $\underline{\mathcal{F}}_{\mathcal{K}}^{\mathfrak{D}_A\text{-an}}$

In Subsection 11.2, we studied the algebraic object $j_{\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ using analytical results. In this subsection, we want to investigate the τ-invariants of $(j_{\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)})^{\mathfrak{D}_A\text{-rig}}$ on the cuspidal affinoids of the standard cover of $\overline{\mathfrak{M}}_{\mathcal{K}}^{\mathrm{rig}}$.

The idea is that at a point z near a fixed cusp one has, due to Drinfeld, an exact sequence

$$0 \longrightarrow \bar{\Lambda} \longrightarrow \mathbb{C}_\infty \xrightarrow{e_{\bar{\Lambda}}} \mathbb{C}_\infty \longrightarrow 0,$$

for a suitable rank 1 lattice $\bar{\Lambda}$ where the group in the middle has the structure of a rank one Drinfeld-module and that on the left the structure of a rank 2 Drinfeld-module. The rank 1 Drinfeld-module extends to the cusp, the rank 2 Drinfeld module degenerates at the cusp. Passing to the associated analytic $\mathfrak{D}_A$-motives yields a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(\bar{\Lambda}, \Omega_A) \otimes_A \tilde{\underline{\mathbb{1}}}_{\mathrm{Spec}\,\mathbb{C}_\infty, \mathfrak{D}_A} \longrightarrow \tilde{\underline{\mathcal{M}}}_{2,z} \longrightarrow \tilde{\underline{\mathcal{M}}}_{1,z} \longrightarrow 0.$$

Here the $\tilde{\underline{\mathcal{M}}}_{i,z}$ are the $\mathfrak{D}_A$-analytic A-motives of rank i attached to the Drinfeld-modules of rank i. The main result of this subsection is to provide a construction which shows that the above short exact sequence exists in a uniform way near any fixed cusp, and hence that the unit-τ-sheaf is a τ-subsheaf of the maximal extension of $\tilde{\underline{\mathcal{M}}}_2$.

By Theorem 4.16, we have $\overline{\mathfrak{M}}_{\mathcal{K}}^{\mathrm{rig}} \cong \amalg_\nu \Gamma_\nu \backslash \bar{\Omega}$ via the isomorphism $\bar{\xi}$, where the ν are representatives of $\mathrm{Cl}_{\mathcal{K}}$. The cuspidal affinoids are most easily described in local notation, i.e., by fixing a ν. As the definition of maximal extension is of a local nature, cf. Proposition 11.45, it, too, can be defined over an individual component. So while we are working in this local setting, let us drop the index ν.

As in Section 3, we fix an arithmetic subgroup Γ of $\mathrm{GL}_2(K)$ which is p'-torsion free (by admissibility of $\mathcal{K}$), and a Γ-stable lattice Λ of K^2 (the lattice Λ_{x_ν} of Section 4). Let $\underline{s}$ be a rational cusp, which, without loss of generality, we assume to be $(0:1)$. We recall some more notation from Section 3, and in particular its last subsection. By $\Omega_{\underline{s}}$, we denote the rigid space $\bigcup_t \mathfrak{U}_t$ where the union is over all simplices in the subtree $\mathcal{T}_{\underline{s}}$ of $\mathcal{T}_\infty$ which belong to the end $\underline{s}$. By $\Gamma_{\underline{s}} \subset \Gamma$, the stabilizer of $\underline{s}$ under Γ is denoted. Because Γ is p'-torsion free, we have

$$\Gamma_{\underline{s}} = \left\{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right) : b \in I_{\underline{s}} \right\}$$

for some fractional almost-ideal $I_{\underline{s}}$. In Lemma 3.31, we observed that, by adding a single puncture to $\Gamma_{\underline{s}} \backslash \Omega_{\underline{s}}$, we obtain an affinoid subdomain $\Gamma_{\underline{s}} \backslash \bar{\Omega}$ of $\mathbb{P}^{1,\mathrm{rig}}_{K_\infty}$.

For every $(y_0, y_1) \in \Lambda$, we defined a section $s_{(y_0,y_1)} \colon \Omega \to \mathbb{C}_\infty : z \mapsto y_0 z + y_1$. Thus we have a map $\Lambda \to \mathcal{O}_\Omega$. For $z \in \Omega$, let Λ_z be the discrete rank 2 lattice of $\mathbb{C}_\infty$ given by $\{y_0 z + y_1 : (y_0, y_1) \in \Lambda\}$. The Drinfeld-module ψ on Ω attached to Λ was defined by

$$\begin{tikzcd} \mathcal{O}_\Omega \arrow[r, "s \mapsto e_\Lambda \circ s"] \arrow[d, "s \mapsto as"'] & \mathcal{O}_\Omega \arrow[d, "s \mapsto \psi_a \circ s"] \\ \mathcal{O}_\Omega \arrow[r, "s \mapsto e_\Lambda \circ s"] & \mathcal{O}_\Omega, \end{tikzcd}$$

where $(e_\Lambda \circ s)(z) = e_{\Lambda_z}(s(z))$.

We set $\Lambda_1 := \Lambda^{\Gamma_{\underline{s}}} = \{(y_0, y_1) \in \Lambda : y_0 = 0\}$, which is a saturated submodule of Λ, and define $\bar{\Lambda} := \Lambda/\Lambda_1$. Furthermore, define $\Lambda_1' := \{y_1 : (0, y_1) \in \Lambda\}$ and $\Lambda_0 := \{y_0 \in K : \exists y_1 \in K : (y_0, y_1) \in \Lambda\}$. The map $(y_0, y_1) \to y_0$, induces an isomorphism $\bar{\Lambda} \cong \Lambda_0$. Because Λ_1 is $\Gamma_{\underline{s}}$-invariant, it gives rise to a local system

$$0 \longrightarrow \Lambda_1' \longrightarrow \mathcal{O}_{\Gamma_s \backslash \Omega_{\underline{s}}}(\Gamma_s \backslash \Omega_{\underline{s}}) \xrightarrow{e_{\Lambda_1'}} \mathcal{O}_{\Gamma_{\underline{s}} \backslash \Omega_{\underline{s}}}(\Gamma_s \backslash \Omega_{\underline{s}}).$$

The exponential $e_{\Lambda_1'}$ defines a rank one Drinfeld-module on $\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}$, which we denote by φ. Note that $\{y_0z+y_1 : (y_0,y_1)\in\Lambda_1\}$ is independent of z, and so there is no need to introduce $\Lambda_{1,z}$.

The sections $s_{(y_0,y_1)}$ induce locally analytic sections of $\mathcal{O}_{\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}}$. However for $y_0\neq 0$ these are not Γ_t-invariant for $t\in\mathcal{T}_{\underline{s}}$, and hence the sections are not rigid analytic. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\Gamma_s\backslash\Omega_{\underline{s}}}(\Gamma_s\backslash\Omega_{\underline{s}}) \arrow[r, "s\mapsto e_{\Lambda_1'}\circ s"] & \mathcal{O}_{\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}}(\Gamma_s\backslash\Omega_{\underline{s}}) \\
\Lambda \arrow[u, "(y_0,y_1)\mapsto s_{y_0,y_1}"] \arrow[r, two heads, "(y_0,y_1)\mapsto y_0"] & \Lambda_0, \arrow[u, "y_0\mapsto(z\mapsto e_{\Lambda_1'}(y_0z))_{z\in\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}}"']
\end{tikzcd}$$

where we claim that for $y_0\in\Lambda_0$ the section $\Omega_{\underline{s}}\to\mathbb{C}_\infty : z\mapsto e_{\Lambda_1'}(y_0z)$ is invariant under the action of $\Gamma_{\underline{s}}$, and hence that the right hand vertical map is well-defined: Fix $\gamma=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\in\Gamma_{\underline{s}}$. Since γ acts on Λ, it follows that for $(y_0,y_1)\in\Lambda$ and $b\in I_{\underline{s}}$ we have $(y_0,-by_0+y_1)\in\Lambda$, and hence by additivity of Λ that $by_0\in\Lambda_1'$. Therefore

$$\begin{aligned}(e_{\Lambda_1'}(y_0\gamma z),\gamma z) &= (e_{\Lambda_1'}(y_0z+by_0),\gamma z)=(e_{\Lambda_1'}(y_0z)+e_{\Lambda_1'}(by_0),\gamma z)\\ &= (e_{\Lambda_1'}(y_0z),\gamma z).\end{aligned}$$

This finishes the proof of the claim.

For given $y_0\in\Lambda_0$, we denote the corresponding section $z\mapsto e_{\Lambda_1'}(y_0z)$ by $\bar{s}_{y_0}$. A straightforward calculation shows that $\varphi_a(\bar{s}_{y_0})=\bar{s}_{ay_0}$. Thus if we have A act on $\mathbb{C}_\infty$ via φ, then for each z the set $\bar{\Lambda}_z:=\{e_{\Lambda_1'}(y_0z) : y_0\in\Lambda\}$ defines a discrete A-lattice of rank one. It is easily checked that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_{\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}} \arrow[r, "s\mapsto e_{\bar{\Lambda}}\circ s"] \arrow[d, "s\mapsto\varphi_a\circ s"] & \mathcal{O}_{\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}} \arrow[d, "s\mapsto\psi_a\circ s"] \\
\mathcal{O}_{\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}} \arrow[r, "s\mapsto e_{\bar{\Lambda}}\circ s"] & \mathcal{O}_{\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}}.
\end{tikzcd}$$

By $\underline{\mathcal{E}}_1$ we denote the A-module corresponding to the rank one Drinfeld-module defined by φ, and by $\underline{\mathcal{E}}_2$ that defined by ψ. The map $(w,z)\mapsto(e_{\bar{\Lambda}_z}(w),z)$ defines a rigid analytic map $\underline{\mathcal{E}}_1\to\underline{\mathcal{E}}_2$ on $\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}$ defined over K_∞. Let $\underline{\tilde{\mathcal{M}}}_1$ denote the rigid A-motive on $\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}$ corresponding to $\underline{\mathcal{E}}_1$ and $\underline{\tilde{\mathcal{M}}}_2$ the one corresponding to $\underline{\mathcal{E}}_2$. Then the above yields a morphism $\underline{\tilde{\mathcal{M}}}_2\to\underline{\tilde{\mathcal{M}}}_1$ in $\widetilde{\mathbf{Coh}}_\tau(\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}},A)$. We define $\underline{\tilde{\mathcal{H}}}$ and $\underline{\tilde{\mathcal{C}}}$ via the four term exact sequence

$$0\longrightarrow\underline{\tilde{\mathcal{H}}}\longrightarrow\underline{\tilde{\mathcal{M}}}_2\longrightarrow\underline{\tilde{\mathcal{M}}}_1\longrightarrow\underline{\tilde{\mathcal{C}}}\longrightarrow 0.$$

The following summarizes the central result of this subsection:

Theorem 11.51 *We have $\underline{\tilde{\mathcal{C}}}=0$ and*

$$\underline{\tilde{\mathcal{H}}}^{\mathfrak{D}_A\text{-rig}}\cong\mathrm{Hom}_A(\bar{\Lambda},\Omega_A)\otimes_A\underline{\tilde{\mathbb{1}}}_{\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}},\mathfrak{D}_A}.$$

As a consequence, we will in the end obtain the following important corollary:

Corollary 11.52 *For all $n\geq 0$, the $\mathfrak{D}_A$-rigid τ-sheaf $\mathrm{Sym}^n\,\underline{\tilde{\mathcal{M}}}_2^{\mathfrak{D}_A\text{-rig}}$ admits a maximal extension under the Zariski-open immersion*

$$j_{\underline{s}}\colon\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}\to\Gamma_{\underline{s}}\backslash\bar{\Omega}_{\underline{s}}.$$

It satisfies $(\mathrm{Sym}^n(\underline{\tilde{\mathcal{M}}}_2^{\mathfrak{D}_A\text{-rig}})^{\max})^\tau\cong\mathrm{Hom}_A(\bar{\Lambda},\Omega_A)^{\otimes n}$.

We first prove $\underline{\tilde{\mathcal{C}}} = 0$. For this we need the following lemma.

Lemma 11.53 *Let $\underline{\tilde{\mathcal{F}}}$ be an A-motive on $\mathbb{C}_\infty$ and*

$$0 \longrightarrow \underline{\tilde{\mathcal{K}}} \longrightarrow \underline{\tilde{\mathcal{F}}} \longrightarrow \underline{\tilde{\mathcal{G}}} \longrightarrow 0$$

a short exact sequence of A-rigid τ-sheaves on $\mathbb{C}_\infty$. Then for any non-zero ideal $\mathfrak{l}$ of A one has a commutative diagram

$$\begin{tikzcd}
\underline{\tilde{\mathcal{K}}}/\mathfrak{l} \arrow[r] \arrow[d] & \underline{\tilde{\mathcal{F}}}/\mathfrak{l} \arrow[r] \arrow[d, "\cong"] & \underline{\tilde{\mathcal{G}}}/\mathfrak{l} \arrow[r] \arrow[d, "\cong"] & 0 \\
(\underline{\tilde{\mathcal{K}}}/\mathfrak{l})^\tau \otimes_k \mathbb{C}_\infty \arrow[r] & (\underline{\tilde{\mathcal{F}}}/\mathfrak{l})^\tau \otimes_k \mathbb{C}_\infty \arrow[r, two heads] & (\underline{\tilde{\mathcal{G}}}/\mathfrak{l})^\tau \otimes_k \mathbb{C}_\infty \arrow[r] & 0,
\end{tikzcd}$$

where the top sequence is right exact, the bottom sequence is exact at $(\underline{\tilde{\mathcal{G}}}/\mathfrak{l})^\tau \otimes \mathbb{C}_\infty$, and the middle and right vertical maps are isomorphisms. If furthermore $\underline{\tilde{\mathcal{G}}}$ is projective over $\mathbb{C}_\infty \otimes_k A$, then both horizontal sequences are short exact and all three vertical maps are isomorphisms.

Proof: By left exactness of $\otimes$, one has a left exact sequence of τ-sheaves

$$\underline{\tilde{\mathcal{K}}}/\mathfrak{l} \longrightarrow \underline{\tilde{\mathcal{F}}}/\mathfrak{l} \longrightarrow \underline{\tilde{\mathcal{G}}}/\mathfrak{l} \longrightarrow 0. \tag{63}$$

Because $\underline{\tilde{\mathcal{F}}}$ is an A-motive, the induced τ is an isomorphism on $\underline{\tilde{\mathcal{F}}}/\mathfrak{l}$. Lang's theorem implies that $(\underline{\tilde{\mathcal{F}}}/\mathfrak{l})^\tau \otimes_k \mathbb{C}_\infty \cong \underline{\tilde{\mathcal{F}}}/\mathfrak{l}$. The analogous statements for $\underline{\tilde{\mathcal{G}}}$ in place of $\underline{\tilde{\mathcal{F}}}$ hold, too.

If $\underline{\tilde{\mathcal{G}}}$ is projective, then the sequence (63) is short exact, and the induced τ on $\underline{\tilde{\mathcal{K}}}/\mathfrak{l}$ is an isomorphism as well. ∎

Proof of $\underline{\tilde{\mathcal{C}}} = 0$: It suffices to show $\underline{\tilde{\mathcal{C}}} = 0$ pointwise for all $z \in \Gamma_{\underline{s}} \backslash \Omega_{\underline{s}}$. Specialized to any such z, one has the short exact sequence

$$0 \longrightarrow \Lambda_0 \longrightarrow \mathbb{C}_\infty \xrightarrow{e_{\bar{\Lambda}_z}} \mathbb{C}_\infty \longrightarrow 0.$$

It yields a short exact sequence of $\mathfrak{l}$-torsion points for any non-zero ideal $\mathfrak{l}$ of A. Interpreted in terms of the associated τ-sheaves, this implies that

$$(\underline{\tilde{\mathcal{M}}}_{2,z}/\mathfrak{l})^\tau \longrightarrow\!\!\!\!\!\rightarrow (\underline{\tilde{\mathcal{M}}}_{1,z}/\mathfrak{l})^\tau$$

is surjective. The above lemma and the definition of $\underline{\tilde{\mathcal{C}}}$ thus imply that $(\underline{\tilde{\mathcal{C}}}_z/\mathfrak{l})^\tau = 0$ for all non-zero ideals $\mathfrak{l}$ of A, and so in particular that $\underline{\tilde{\mathcal{C}}}_z/\mathfrak{l} = 0$ for all prime ideals $\mathfrak{l}$ of A. Therefore $\operatorname{Supp} \underline{\tilde{\mathcal{C}}}_z \subset \operatorname{Spec} \mathbb{C}_\infty \otimes_k A$ is finite and disjoint from $\{\mathbb{C}_\infty \otimes \mathfrak{l} : \mathfrak{l} \text{ a prime ideal of } A\}$.

The epimorphism $\underline{\tilde{\mathcal{M}}}_1 \longrightarrow\!\!\!\!\!\rightarrow \underline{\tilde{\mathcal{C}}}$ induces a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \sigma^*_{\mathbb{C}_\infty, A} \underline{\tilde{\mathcal{M}}}_{1,z} \arrow[r] \arrow[d, two heads] & \underline{\tilde{\mathcal{M}}}_{1,z} \arrow[r] \arrow[d, two heads] & \underline{\tilde{\mathcal{C}}}'_z \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \sigma^*_{\mathbb{C}_\infty, A} \underline{\tilde{\mathcal{C}}}_z \arrow[r] & \underline{\tilde{\mathcal{C}}}_z \arrow[r] & \underline{\tilde{\mathcal{C}}}''_z \arrow[r] & 0,
\end{tikzcd}$$

where $\underline{\tilde{\mathcal{C}}}'_z$ and $\underline{\tilde{\mathcal{C}}}''_z$ are defined by the diagram. The surjectivity on the right follows from the snake lemma. Because $\underline{\tilde{\mathcal{C}}}'_z$ is supported on the prime ideal generated by $\iota(a) \otimes 1 - 1 \otimes a$, so must be $\underline{\tilde{\mathcal{C}}}''_z$.

Now $\sigma_{\mathbb{C}_\infty,A}$ acts on the prime ideals of $\mathbb{C}_\infty \otimes_k A$ and the only prime ideals with finite orbits are those of the form $\mathbb{C}_\infty \otimes \mathfrak{l}$ where $\mathfrak{l}$ is a prime ideal of A. This shows that $\underline{\tilde{\mathcal{C}}}$ can only have finite support if $\underline{\tilde{\mathcal{C}}}''_z = 0$. But in this case $\sigma^*_{\mathbb{C}_\infty,A}\underline{\tilde{\mathcal{C}}}_z \longrightarrow \underline{\tilde{\mathcal{C}}}_z$ is an isomorphism. By the same reasoning it follows that $\underline{\tilde{\mathcal{C}}}_z$ has its support in the set of primes $\mathbb{C}_\infty \otimes \mathfrak{l}$, and by the above $\underline{\tilde{\mathcal{C}}}_z = 0$. ∎

Above we defined A-modules $\underline{\mathcal{E}}_i$, $i = 1, 2$ over $\Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}$. Pulling these back along $\pi_{\underline{s}} : \Omega_{\underline{s}} \to \Gamma_{\underline{s}}\backslash\Omega_{\underline{s}}$, we obtain A-modules on $\Omega_{\underline{s}}$, which we denote by $\underline{\mathcal{E}}'_i$. The corresponding A-rigid τ-sheaves are called $\underline{\tilde{\mathcal{M}}}'_i$. Furthermore by $\underline{\tilde{\mathcal{H}}}'$, we denote the kernel of $\underline{\tilde{\mathcal{M}}}'_2 \to \underline{\tilde{\mathcal{M}}}'_1$. On the standard affinoids of $\Omega_{\underline{s}}$, the map $\pi_{\underline{s}}$ is finite flat, and hence $\underline{\tilde{\mathcal{H}}}'$ is isomorphic to the pullback of $\underline{\tilde{\mathcal{H}}}$ along $\pi_{\underline{s}}$, and we have a short exact sequence

$$0 \longrightarrow \underline{\tilde{\mathcal{H}}}' \longrightarrow \underline{\tilde{\mathcal{M}}}'_2 \longrightarrow \underline{\tilde{\mathcal{M}}}'_1 \longrightarrow 0. \tag{64}$$

Let us apply the left exact functor $\mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Omega_{\underline{s}},\mathfrak{D}_A)}(\_\_, \underline{\tilde{\mathbb{1}}}_{\Omega_{\underline{s}},\mathfrak{D}_A} \otimes_A \Omega_A)$ to the above sequence, and make use of the results of Subsection 9.2. We obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Lambda_1 \cong \Gamma(\Omega_{\underline{s}}, H_*(\underline{\mathcal{E}}'_1)) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Omega_{\underline{s}},\mathfrak{D}_A)}(\underline{\tilde{\mathcal{M}}}'_1, \underline{\tilde{\mathbb{1}}}_{\Omega_{\underline{s}},\mathfrak{D}_A} \otimes_A \Omega_A) \arrow[d] \\
\Lambda \cong \Gamma(\Omega_{\underline{s}}, H_*(\underline{\mathcal{E}}'_2)) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Omega_{\underline{s}},\mathfrak{D}_A)}(\underline{\tilde{\mathcal{M}}}'_2, \underline{\tilde{\mathbb{1}}}_{\Omega_{\underline{s}},\mathfrak{D}_A} \otimes_A \Omega_A) \arrow[d] \\
\bar{\Lambda} \arrow[r] \arrow[d] & \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Omega_{\underline{s}},\mathfrak{D}_A)}(\underline{\tilde{\mathcal{H}}}', \underline{\tilde{\mathbb{1}}}_{\Omega_{\underline{s}},\mathfrak{D}_A} \otimes_A \Omega_A) \\
0, &
\end{tikzcd} \tag{65}$$

where the columns as displayed are exact, and the top and middle horizontal maps are isomorphisms. Therefore the bottom horizontal map must be a monomorphism.

Lemma 11.54 *The bottom horizontal map in (65) is an isomorphism.*

Proof: It suffices to prove the lemma for the fiber at any closed point, say $z \in \Omega_{\underline{s}}$. Because the sheaves $\tilde{\mathcal{M}}'_i$ are locally free of rank i, the sheaf $\tilde{\mathcal{H}}'$ must be free of rank 1. Hence by [1], Lem. 2.10.6, the fiber $\underline{\tilde{\mathcal{H}}}'_z$ at z is isomorphic to $\mathrm{Hom}_A(P, \Omega_A) \otimes \underline{\tilde{\mathbb{1}}}_{\mathbb{C}_\infty,A}$ for some projective A-module P of rank one. We need to show that the map $\bar{\Lambda} \to P$ obtained from the above diagram is an isomorphism.

The fiber at z of the short exact sequence (64) is isomorphic to the tensor product over k of the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(\Lambda_1, \Omega_A) \longrightarrow \mathrm{Hom}_A(\Lambda, \Omega_A) \longrightarrow \mathrm{Hom}_A(P, \Omega_A) \longrightarrow 0$$

with $\underline{\tilde{\mathbb{1}}}_{\mathbb{C}_\infty,A}$. Computing $\mathfrak{l}$-torsion points for any non-zero ideal $\mathfrak{l}$ of A yields

$$0 \longrightarrow \mathrm{Hom}_A(\Lambda_1, \Omega_A)/\mathfrak{l} \longrightarrow \mathrm{Hom}_A(\Lambda, \Omega_A)/\mathfrak{l} \longrightarrow \mathrm{Hom}_A(P, \Omega_A)/\mathfrak{l} \longrightarrow 0. \tag{66}$$

But the above exact sequence of $\mathfrak{l}$-torsion points arises from the left exact sequence

$$0 \to \Lambda_0 \to \underline{\mathcal{E}}'_{1,z} \to \underline{\mathcal{E}}'_{2,z}, \tag{67}$$

and hence it follows that the cokernel of the sequence (66) is isomorphic to $\mathrm{Hom}_A(\bar{\Lambda}, \Omega_A)/\mathfrak{l}$. One concludes that for each $\mathfrak{l}$ the reduction modulo $\mathfrak{l}$ of the map $\bar{\Lambda} \to P$ is an isomorphism. This proves the lemma. ∎

We now need the following rather technical, if simple, result.

Lemma 11.55 *Diagram (65) is compatible with the action of* $\Gamma_{\underline{s}}$.

PROOF: The sequence (64) is induced from the left exact sequence (67). As the latter is $\Gamma_{\underline{s}}$-equivariant, so is the former, and hence all the maps in the right column are $\Gamma_{\underline{s}}$-equivariant.

The $\Gamma_{\underline{s}}$-equivariance of the left hand column is rather trivial, and so to prove the lemma, it suffices to show that the top and middle horizontal maps are $\Gamma_{\underline{s}}$-equivariant. We will only give the proof for $\underline{\mathcal{E}}'_2$, the proof for $\underline{\mathcal{E}}'_1$ being analogous.

Based on ideas of Anderson, the middle horizontal isomorphism was constructed in Section 9 as the following composite of morphisms

$$\begin{aligned}
\Lambda \;&\xrightarrow{\cong}\; \mathrm{Hom}^c_A(K_\infty/A, \underline{\mathcal{E}}'_2) && (68)\\
&\xrightarrow{\cong}\; \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Omega_{\underline{s}},A)}(\underline{\mathcal{M}}(\mathcal{E}'_2), \mathrm{Hom}^c(K_\infty/A, \mathcal{O}_{\Omega_{\underline{s}}})) && (69)\\
&\xrightarrow{\cong}\; \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Omega_{\underline{s}},\mathfrak{D}_A)}(\underline{\tilde{\mathcal{M}}}'_2, \mathrm{Hom}^c(K_\infty/A, \underline{\tilde{\mathbb{1}}}_{\Omega_{\underline{s}},\mathfrak{D}_A} \otimes_A \Omega_A)), && (70)
\end{aligned}$$

where one used the $\Gamma_{\underline{s}}$-equivariant identification

$$\mathrm{Hom}^c(K_\infty/A, \mathcal{O}_{\Omega_{\underline{s}}}) \cong \underline{\tilde{\mathbb{1}}}_{\Omega_{\underline{s}},\mathfrak{D}_A} \otimes_A \Omega_A$$

provided by Lemma 9.28. The isomorphism in line (70) is easily seen to be $\Gamma_{\underline{s}}$-equivariant. Let us denote the isomorphisms in lines (68) and (69) by α and β, respectively.

We first show that β is $\Gamma_{\underline{s}}$-equivariant. For this, we may work on a standard affinoid Ω_t of $\Omega_{\underline{s}}$ and fix $\gamma \in \Gamma_{\underline{s}}$. The element γ acts directly on $\underline{\mathcal{E}}'_2$, on $\underline{\mathcal{M}}(\mathcal{E}'_2) \cong \mathrm{Hom}(\underline{\mathcal{E}}'_2, \mathbb{G}_a)^{\mathfrak{D}_A\text{-rig}}$ and on $\mathcal{O}_{\Omega_{\underline{s}}}$, and in the usual way on modules of homomorphism between these objects and objects with trivial $\Gamma_{\underline{s}}$-action. For any

$$\begin{aligned}
f &\in \mathrm{Hom}^c_A(K_\infty/A, \Gamma(\Omega_{\underline{s}}, \underline{\mathcal{E}}'_2))\\
m &\in \Gamma(\Omega_{\underline{s}}, \underline{\tilde{\mathcal{M}}}'_2) \text{ and}\\
c &\in K_\infty/A,
\end{aligned}$$

we have

$$f \overset{\beta}{\mapsto} (m \mapsto (c \mapsto m(f(c)))),$$

and this characterizes β. Since $(\gamma f) = \gamma \circ f$, $(\gamma m) = \gamma \circ m \circ \gamma^{-1}$, it follows that

$$\begin{aligned}
(\beta(\gamma f)m)(c) &= (m \circ \gamma \circ f)(c) = \gamma(((\gamma^{-1}m) \circ f)(c))\\
&= \gamma((\beta(f)(\gamma^{-1}m))(c)),
\end{aligned}$$

i.e., that $\beta(\gamma f) = \gamma(\beta f)$, as asserted.

To see that α is $\Gamma_{\underline{s}}$-equivariant, we consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\alpha"', "\cong"] & \mathrm{Lie}(\underline{\mathcal{E}}_2') \arrow[r] \arrow[d, "\delta"', "\cong"] & \underline{\mathcal{E}}_2' \arrow[d, "\varepsilon"', "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_A^c(K_\infty/A, \underline{\mathcal{E}}_2') \arrow[r] & \mathrm{Hom}_A^c(K_\infty, \underline{\mathcal{E}}_2') \arrow[r] & \mathrm{Hom}_A^c(A, \underline{\mathcal{E}}_2'),
\end{tikzcd}$$

where clearly both rows are $\Gamma_{\underline{s}}$-equivariant, and so is the right vertical isomorphism. Thus to show that α is $\Gamma_{\underline{s}}$-equivariant, it suffices to show this for δ. For the latter we have the explicit formula (46) in the proof of Proposition 9.2, namely

$$\mathrm{Lie}(\underline{\mathcal{E}}_2') \to \mathrm{Hom}_A^c(K_\infty, \underline{\mathcal{E}}_2') : x \mapsto (\xi_x : y \mapsto \exp_{\underline{\mathcal{E}}_2'}(yx)).$$

Since γ acts trivially on K_∞, we have

$$\xi_{\gamma x}(y) = \exp_{\underline{\mathcal{E}}_2'}(y\gamma x) = \exp_{\underline{\mathcal{E}}_2'}(\gamma(yx)) = \gamma(\exp_{\underline{\mathcal{E}}_2'}(yx)) = \gamma(\xi_x(y)).$$

This finally concludes the proof of Lemma 11.55. ∎

Observe now that $\bar{\Lambda}$ is $\Gamma_{\underline{s}}$-invariant. Therefore the above lemma yields a monomorphism

$$\bar{\Lambda} \hookrightarrow \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Omega_{\underline{s}}, \mathfrak{D}_A)}((\underline{\tilde{\mathcal{H}}}')^{\Gamma_{\underline{s}}}, (\underline{\tilde{\mathbb{1}}}_{\Omega_{\underline{s}}, \mathfrak{D}_A} \otimes_A \Omega_A)^{\Gamma_{\underline{s}}}).$$

Since $(\underline{\tilde{\mathcal{M}}}_i')^{\Gamma_{\underline{s}}} \cong \underline{\tilde{\mathcal{M}}}_i$ for $i = 1, 2$, we have that $(\underline{\tilde{\mathcal{H}}}')^{\Gamma_{\underline{s}}} \cong \underline{\tilde{\mathcal{H}}}$. Furthermore it is rather obvious that

$$(\underline{\tilde{\mathbb{1}}}_{\Omega_{\underline{s}}, \mathfrak{D}_A} \otimes_A \Omega_A)^{\Gamma_{\underline{s}}} \cong \underline{\tilde{\mathbb{1}}}_{\Gamma_{\underline{s}} \backslash \Omega_{\underline{s}}, \mathfrak{D}_A} \otimes_A \Omega_A.$$

Thus we have a monomorphism

$$\bar{\Lambda} \overset{\alpha'}{\hookrightarrow} \mathrm{Hom}_{\widetilde{\mathbf{Coh}}_\tau(\Gamma_{\underline{s}} \backslash \Omega_{\underline{s}}, \mathfrak{D}_A)}(\underline{\tilde{\mathcal{H}}}, \underline{\tilde{\mathbb{1}}}_{\Gamma_{\underline{s}} \backslash \Omega_{\underline{s}}, \mathfrak{D}_A} \otimes_A \Omega_A).$$

In the proof of Lemma 11.54, we proved that the specialization to any fiber is an isomorphism. Therefore we have shown:

Corollary 11.56 *The map α' is an isomorphism.*

Proof of Theorem 11.51: To conclude the proof of the theorem, it remains to show

$$\underline{\tilde{\mathcal{H}}}^{\mathfrak{D}_A\text{-rig}} \cong \mathrm{Hom}_A(\bar{\Lambda}, \Omega_A) \otimes_A \underline{\tilde{\mathbb{1}}}_{\Gamma_{\underline{s}} \backslash \Omega_{\underline{s}}, \mathfrak{D}_A}.$$

However this is immediate from the previous corollary and Theorem 9.19. ∎

Proof of Corollary 11.52: As a consequence of the above corollary and Corollary 11.49, we find that

$$j_{\underline{s}\#}\underline{\tilde{\mathcal{H}}}^{\mathfrak{D}_A\text{-rig}} \cong \mathrm{Hom}_A(\bar{\Lambda}, \Omega_A) \otimes_A \underline{\tilde{\mathbb{1}}}_{\Gamma_{\underline{s}} \backslash \bar{\Omega}_{\underline{s}}, \mathfrak{D}_A} \subset j_{\underline{s}\#}\underline{\tilde{\mathcal{M}}}_2^{\mathfrak{D}_A\text{-rig}}.$$

Because by Proposition 11.30 $j_{\mathcal{K}\#}\mathfrak{b}_{K_\infty}^*\underline{\mathcal{F}}_{\mathcal{K}}$ is locally free of rank 2, Proposition 11.50 shows that the sheaf underlying $j_{\underline{s}\#}\underline{\tilde{\mathcal{M}}}_2$ is a locally free rigid sheaf of rank 2. Passing to symmetric powers, we obtain

$$\left(j_{\underline{s}\#}\underline{\tilde{\mathcal{H}}}^{\mathfrak{D}_A\text{-rig}}\right)^{\otimes n} \cong \mathrm{Hom}_A(\bar{\Lambda}, \Omega_A)^{\otimes n} \otimes_A \underline{\tilde{\mathbb{1}}}_{\Gamma_{\underline{s}} \backslash \bar{\Omega}_{\underline{s}}, \mathfrak{D}_A} \subset j_{\underline{s}\#} \operatorname{Sym}^n \underline{\tilde{\mathcal{M}}}_2^{\mathfrak{D}_A\text{-rig}}$$

for any $n \geq 0$. Combining Proposition 11.50 and Theorem 11.30 shows that $i^*\underline{\tilde{\mathcal{C}}}_n$ is nilpotent for the cokernel $\underline{\tilde{\mathcal{C}}}_n$ of the above inclusion (this also uses that $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^{\mathfrak{D}_A\text{-rig}}$ preserves exact sequences). Consider now the left exact sequence

$$0 \longrightarrow ((j_{\underline{s}\#}\underline{\tilde{\mathcal{H}}}^{\mathfrak{D}_A\text{-rig}})^{\otimes n})^\tau \cong \mathrm{Hom}_A(\bar{\Lambda}, \Omega_A)^{\otimes n} \longrightarrow (j_{\underline{s}\#} \operatorname{Sym}^n \underline{\tilde{\mathcal{M}}}_2)^\tau \longrightarrow (\underline{\tilde{\mathcal{C}}}_n)^\tau.$$

By Lemma 10.4, the right hand term vanishes, and thus we have completed the proof of the corollary. ∎

12 An Eichler-Shimura isomorphism for double cusp forms

The analogue of Theorem 10.3 for double cusp forms is formally obtained by replacing $j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ by $j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ and the module of integral cusp forms by that of integral double cusp forms. The main new ingredient compared to the proof of Theorem 10.3 is the evaluation of $(j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)})^{\mathfrak{D}_A\text{-rig}}$ near the cusps, which was carried out in the Subsection 11.5.

Definition 12.1 *The* A-crystal of Drinfeld double cusp forms of weight $n+2$ and and level $\mathcal{K}$ *is defined as*

$$\underline{\mathcal{S}}_{n+2}(\mathcal{K}) := \bar{g}_{\mathcal{K}*}j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}.$$

We define the rigid $\mathfrak{D}_A$-crystal on $\operatorname{Spm} K_\infty$ of Drinfeld double cusp forms of weight n and level $\mathcal{K}$ *as*

$$\underline{\mathcal{S}}_n^{2,\mathfrak{D}_A\text{-rig}}(\mathcal{K}) := \underline{\mathcal{S}}_n^2(\mathcal{K})^{\mathfrak{D}_A\text{-rig}}.$$

Let v be a place of K. We define the constructible étale v-adic sheaf on $\operatorname{Spec} A(\mathfrak{n})$ of Drinfeld double cusp forms of weight n and level $\mathcal{K}$ *as*

$$\underline{\mathcal{S}}_n^{2,\text{ét},v}(\mathcal{K}) := \varprojlim_m (\underline{\mathcal{S}}_n^2(\mathcal{K})/\mathfrak{p}_v^m\underline{\mathcal{S}}_n^2(\mathcal{K}))_{\text{ét}}.$$

In general the formation of maximal extension does not commute with pullbacks. However for the base change from A to K_∞ and the τ-sheaf $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ it does by Theorem 11.30. The formation of maximal extension is also known to commute with analytification, cf. Proposition 11.50. If we define $\underline{\tilde{\mathcal{F}}}_{\mathcal{K}}^{(n)} := \operatorname{Sym}^n(\underline{\tilde{\mathcal{F}}}_{\mathcal{K}}^{\mathfrak{D}_A\text{-rig}}) \in \widetilde{\mathbf{Coh}}_\tau(\mathfrak{M}_{\mathcal{K}}^{\text{rig}}, \mathfrak{D}_A)$ (note that this is a slight change of conventions, as $\underline{\tilde{\mathcal{F}}}_{\mathcal{K}}$ is a τ-sheaf over A), then we have shown the following compatibility:

Proposition 12.2 $\underline{\mathcal{S}}_{n+2}^{2,\mathfrak{D}_A\text{-rig}}(\mathcal{K}) \cong H^1(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty}^{\text{rig}}, j_{\mathcal{K}\#}\underline{\tilde{\mathcal{F}}}_{\mathcal{K}}^{(n)}))$.

Note also that by Corollary 11.40, there is a canonical isomorphism

$$\underline{\mathcal{S}}_{n+2}^{2,\text{ét},v}(\mathcal{K}) \cong R^1_{\text{ét}}\bar{g}_{\mathcal{K}*}(j_{\mathcal{K}*}\operatorname{Sym}^n\underline{\mathcal{F}}_{\mathcal{K}}^{\text{ét},v}).$$

Theorem 12.3 *For each admissible $\mathcal{K}$, there is an isomorphism*

$$\left(\mathbf{C}_n^{\text{St},2}(\mathcal{K}, A)\right)^* \cong \left(\underline{\mathcal{S}}_n^{2,\mathfrak{D}_\mathrm{A}-\text{rig}}(\mathcal{K})\right)^\tau.$$

Proof: As in the proof of Theorem 10.3, we denote by $\bar{M}$ the local system of $\mathrm{GL}_2(\mathbb{A}_f)$-modules defined by

$$M_g := (\underline{\mathcal{F}}_{\mathcal{K}}^{(n-2),\mathfrak{D}_A\text{-rig}})^\tau_{|\Omega_g} \overset{\text{Cor. 9.20}}{\cong} \operatorname{Sym}^{n-2}(\operatorname{Hom}(\Lambda_g, \Omega_A)) \text{ for } g \in \mathrm{GL}_2(\mathbb{A}^f).$$

Consider the following complex $\overline{\mathcal{C}}_{\mathcal{K},\bullet}^{\text{st},2}$

$$\begin{aligned}\mathbb{Z}[\mathcal{T}_{\mathcal{K},1}^{\text{st},o}] \longrightarrow \mathbb{Z}[\mathcal{T}_{\mathcal{K},0}^{\text{st}}] \oplus \mathbb{Z}[\mathcal{T}_{\mathcal{K},1}^{\text{st}}] \oplus \mathbb{Z}[\mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}] :\\ \tilde{e} \mapsto -[t(\tilde{e})] + [\overline{\tilde{e}}] + \mathbf{b}_{\mathcal{K}}(t(e)),\end{aligned}$$

where we recall that

(i) $t(\tilde{e})$ is the target of the oriented edge $\tilde{e}$,

(ii) $[t(\tilde{e})]$ is zero of $t(\tilde{e})$ is unstable and the symbol for $t(\tilde{e})$ if it is stable, and

(iii) $\mathbf{b}_{\mathcal{K}}$ is the map that sends a vertex $\tilde{v}$ to zero, if $\tilde{v}$ is stable, and to the unique rational end in $\mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$ whose stabilizer contains that of $\tilde{v}$, if $\tilde{v}$ is unstable.

By Definitions 5.44 and 5.17, and Propositions 5.15 and Lemma 5.47, we have

$$H^1\Big(\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st},2}_{\mathcal{K},\bullet}, \bar{M})\Big) \cong \Big(\mathbf{C}^{\mathrm{St},2}_n(\mathcal{K}, A)\Big)^*.$$

Let us choose representatives $R^?_?$ of the stable edges and vertices as in the proof of Theorem 10.3. Let us furthermore choose a set of representatives $R_{c,\nu}$ of the cusps of the component ν, i.e., of representatives of rational ends $[\underline{s}] \in \mathrm{GL}_2(K)\backslash(\mathbb{P}^1(K) \times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K})$ in the component ν. For each oriented edge $e \in R^o_{1,\nu}$ with target $t(e)$, we have a unique vertex $v_e \in R_{0,\nu}$ and a unique $\gamma_e \in \Gamma_\nu$ such that $t(e) = \gamma_e v_e$. Similarly, we define $\gamma'_e \in \Gamma_\nu$ and $[\underline{s}_e] \in R_{c,\nu}$ such that $\mathbf{b}_{\mathcal{K}}(t(e)) = \gamma'_e[\underline{s}_e]$, where the symbol $[\underline{s}_e]$ is zero whenever $t(e)$ is stable.

With all this notation, we have the following explicit description of the complex $\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st},2}_{\mathcal{K},\bullet}, \bar{M})$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},0}, \bar{M}) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathrm{GL}_2(K)}(\overline{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},1}, \bar{M}) \arrow[d, "\cong"] \\
\bigoplus_\nu \Big(\bigoplus_{v \in R_{0,\nu}} M_{x_\nu} \oplus \bigoplus_{v \in R_{1,\nu}} M_{x_\nu} \oplus \bigoplus_{[\underline{s}] \in R_{c,\nu}} (M_{x_\nu})^{\Gamma_{[\underline{s}]}} \Big) \arrow[r] & \bigoplus_\nu \bigoplus_{e \in R^e_{1,\nu}} M_{x_\nu}.
\end{tikzcd}$$

If f_1 is the image of f_0 under the boundary map, then

$$f_1([\tilde{e}]) = f_0([\bar{\tilde{e}}]) - f_0([t(\tilde{e})]) + f_0(\mathbf{b}_{\mathcal{K}}(t(e))) = f_0([\bar{e}]) - \gamma_e f_0([v_e]) + \gamma'_e f_0([\underline{s}_e]).$$

We recall that $[v_e]$ is zero if $t(e)$ is unstable, while $[\underline{s}_e]$ is zero if $t(e)$ is stable.

Let us now turn toward the Čech complex of $(\underline{\mathcal{F}}_{\mathcal{K}})^{\mathfrak{D}_A\text{-rig}}$. Write $\underline{\tilde{\mathcal{F}}}^{(n)}(\mathcal{K})$ for the τ-sheaf $\mathrm{Sym}^n\, \underline{\tilde{\mathcal{F}}}(\mathcal{K})^{\mathfrak{D}_A\text{-rig}}$ on $\Omega_{\mathcal{K}}$ over $\mathfrak{D}_A$ (this is a slight change of conventions, as $\underline{\tilde{\mathcal{F}}}(\mathcal{K})$ is a τ-sheaf over A.) By an argument as in the proof of the first assertion of Corollary 5.22, one has the following identifications:

$$\begin{aligned}
M_{x_\nu}^{\Gamma_{[\underline{s}]}} &= \mathrm{Sym}^{n-2}(\mathrm{Hom}(\Lambda_{x_\nu}, \Omega_A))^{\Gamma_{[\underline{s}]}} = \mathrm{Sym}^{n-2}(\mathrm{Hom}(\Lambda_{x_\nu}, \Omega_A)^{\Gamma_{[\underline{s}]}}) \\
&= \mathrm{Sym}^{n-2}(\mathrm{Hom}((\Lambda_{x_\nu})_{\Gamma_{[\underline{s}]}}, \Omega_A)) = \mathrm{Hom}((\Lambda_{x_\nu})_{\Gamma_{[\underline{s}]}}, \Omega_A)^{\otimes(n-2)} \\
&\overset{\text{Cor. 11.52}}{\cong} ((j_{\mathcal{K}\#}\underline{\tilde{\mathcal{F}}}^{(n-2)}(\mathcal{K})\Big|_{\mathfrak{U}_{[\underline{s}]}})^\tau,
\end{aligned}$$

where $\mathfrak{U}_{[\underline{s}]}$ is the cuspidal affinoid of $\bar{\mathfrak{U}}_{\mathcal{K}}$ for the rational end $[\underline{s}] \in R_{c,\nu}$.

As in the proof of Theorem 10.3, one can describe the standard affinoid cover $\bar{\mathfrak{U}}_{\mathcal{K}}$ in terms of that of $\Omega_{\mathcal{K}}$ and an identification of the cuspidal affinoids with sets $\Gamma_{[\underline{s}]}\backslash\bar{\Omega}$. (The latter was not necessary in loc. cit. as there the τ-invariants vanished near the cusps.) This can again be made explicit using the sets $R^?_?$. Using the definition of M_g and the identification in the previous paragraph, it follows easily that the second row of the above diagram is isomorphic to:

$$\mathcal{C}^0(\bar{\mathfrak{U}}_{\mathcal{K}}, j_{\mathcal{K}\#}\underline{\tilde{\mathcal{F}}}^{(n-2)}_{\mathcal{K}})^\tau \longrightarrow \mathcal{C}^1(\bar{\mathfrak{U}}_{\mathcal{K}}, j_{\mathcal{K}\#}\underline{\tilde{\mathcal{F}}}^{(n-2)}_{\mathcal{K}})^\tau.$$

We need to show that the cokernel of the above morphism is isomorphic to $(H^1(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty}, j_\#\underline{\mathcal{F}}^{(n)}_{\mathcal{K}})^{\mathfrak{D}_A\text{-rig}})^\tau$. From Theorem 11.19, we have the short exact sequence

$$0 \longrightarrow j_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K}} \longrightarrow j_{\mathcal{K}\#}\underline{\mathcal{F}}^{(n)}_{\mathcal{K}} \longrightarrow i_{\mathcal{K}*}\underline{\mathcal{F}}^{(n),\infty}_{\mathcal{K}} \longrightarrow 0. \tag{71}$$

Let us consider the following commutative diagram with exact rows and columns which displays the relationship of the three Čech complex on $\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}$ with respect to the cover $\overline{\mathfrak{U}}_{\mathcal{K}}$. (We abbreviate $\tilde{\underline{\mathcal{F}}}^{(n),\infty}_{\mathcal{K}} := \underline{\mathcal{F}}^{(n),\infty,\mathfrak{D}_A\text{-rig}}_{\mathcal{K}}$ and write j for the open immersion $\mathfrak{M}_{\mathcal{K},K_\infty} \hookrightarrow \overline{\mathfrak{M}}_{\mathcal{K},K_\infty}$ (and its rigid analogue) and i for the closed immersion of the complement $\mathfrak{M}^{\infty}_{\mathcal{K},K_\infty}$.)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & H^0(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}, j_\# \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}}) \arrow[r] \arrow[d] & H^0(\mathfrak{M}^{\infty,\mathrm{rig}}_{\mathcal{K},K_\infty}, \tilde{\underline{\mathcal{F}}}^{(n),\infty}_{\mathcal{K}}) \arrow[d] & \\
0 \arrow[r] & \mathcal{C}^0(\overline{\mathfrak{U}}_{\mathcal{K}}, j_! \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}}) \arrow[r] \arrow[d] & \mathcal{C}^0(\overline{\mathfrak{U}}_{\mathcal{K}}, j_\# \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}}) \arrow[r] \arrow[d] & \mathcal{C}^0(\overline{\mathfrak{U}}_{\mathcal{K}}, i_* \tilde{\underline{\mathcal{F}}}^{(n),\infty}_{\mathcal{K}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C}^1(\overline{\mathfrak{U}}_{\mathcal{K}}, j_! \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}}) \arrow[r] \arrow[d] & \mathcal{C}^1(\overline{\mathfrak{U}}_{\mathcal{K}}, j_\# \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}}) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & H^1(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}, j_! \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}}) \arrow[r] \arrow[d] & H^1(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}, j_\# \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}}) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0. & &
\end{tikzcd}$$

Let $\delta\colon H^0(\mathfrak{M}^{\infty,\mathrm{rig}}_{\mathcal{K},K_\infty}, \tilde{\underline{\mathcal{F}}}^{(n),\infty}_{\mathcal{K}}) \longrightarrow H^1(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}, j_! \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}})$ denote the connecting homomorphism given by the Snake Lemma. Let us take τ–invariants of this diagram. Under this process, the left hand column stays exact by the proof of Theorem 10.3. Furthermore, the second row also stays exact, because after taking τ-invariants, one has an isomorphism between the left and middle terms over all affiniods $\mathfrak{U}_t$ for which t is not a cusp, and an isomorphism between the middle and right term over the remaining affinoids. It follows from the Snake Lemma that one has an isomorphism between

$$\mathrm{Coker}\left(\mathcal{C}^0(\bar{\mathfrak{U}}_{\mathcal{K}}, j_\# \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \longrightarrow \mathcal{C}^1(\bar{\mathfrak{U}}_{\mathcal{K}}, j_\# \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau\right) \text{ and}$$

$$\mathrm{Coker}\left(H^0(\mathfrak{M}^{\infty,\mathrm{rig}}_{\mathcal{K},K_\infty}, \tilde{\underline{\mathcal{F}}}^{(n),\infty}_{\mathcal{K}})^\tau \xrightarrow{\ \delta\ } \left(H^1(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}, j_! \tilde{\underline{\mathcal{F}}}^{(n)}_{\mathcal{K}})\right)^\tau\right).$$

We now invoke the following simple lemma, whose proof is left to the reader and can be obtained by choosing 'good' representatives of the crystals involved.

Lemma 12.4 *Let L be a complete valued field such that $K_\infty \subset L \subset \mathbb{C}_\infty$. Suppose*

$$0 \longrightarrow \underline{\mathcal{F}}' \longrightarrow \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{F}}'' \longrightarrow 0$$

is a short exact sequence in $\mathbf{Crys}(\operatorname{Spec} L, A)$. *Suppose $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$ are locally free and $\underline{\mathcal{F}}$ is uniformizable. Then $\underline{\mathcal{F}}'$ is locally free, $\underline{\mathcal{F}}'$ and $\underline{\mathcal{F}}''$ are uniformizable, and the sequence*

$$0 \longrightarrow (\underline{\mathcal{F}}'^{\mathfrak{D}_A\text{-rig}})^\tau \longrightarrow (\underline{\mathcal{F}}^{\mathfrak{D}_A\text{-rig}})^\tau \longrightarrow (\underline{\mathcal{F}}''^{\mathfrak{D}_A\text{-rig}})^\tau \longrightarrow 0$$

is short exact.

By Corollary 10.13, the crystal $H^1(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty}, j_! \underline{\mathcal{F}}^{(n)}_{\mathcal{K}})$ is uniformizable, and it is rather straightforward, to see that $H^0(\mathfrak{M}^{\infty}_{\mathcal{K},K_\infty}, \mathfrak{b}^*_{K_\infty} \underline{\mathcal{F}}^{(n),\infty}_{\mathcal{K}})$ is uniformizable as well. Also both of these are locally free. We want to apply the lemma to the 4-term sequence that arises from the above diagram by the Snake Lemma. For this, it remains to show that $H^1(\overline{\mathfrak{M}}_{\mathcal{K},K_\infty}, j_\# \underline{\mathcal{F}}_{\mathcal{K}}^{(n)})$ is locally free. We first note that $j_{\mathcal{K}\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(n-2)}$ is a locally free τ-sheaf. It follows that its absolute cohomology $\underline{\mathcal{S}}_n^2(\mathcal{K})$ is a *flat* A-crystal in the terminology of [4], Ch. 6. Because A is a Dedekind domain, it follows by op. cit. that $\mathfrak{b}_K^* \underline{\mathcal{S}}_n^2(\mathcal{K})$ is representable by a τ-module which is projective over $K \otimes A$.

The lemma now yields the following four term exact sequence

$$0 \longrightarrow H^0(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}, j_\# \underline{\tilde{\mathcal{F}}}_{\mathcal{K}}^{(n)})^\tau \longrightarrow H^0(\mathfrak{M}^{\infty,\mathrm{rig}}_{\mathcal{K},K_\infty}, \underline{\tilde{\mathcal{F}}}_{\mathcal{K}}^{(n),\infty})^\tau \longrightarrow$$

$$H^1(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}, j_! \underline{\tilde{\mathcal{F}}}_{\mathcal{K}}^{(n)})^\tau \longrightarrow H^1(\overline{\mathfrak{M}}^{\mathrm{rig}}_{\mathcal{K},K_\infty}, j_\# \underline{\tilde{\mathcal{F}}}_{\mathcal{K}}^{(n)})^\tau \longrightarrow 0$$

This completes the proof of the theorem. ∎

For later use, and the application below, we display the following 4-term exact sequence, which is obtained from (71) by applying $\bar{g}_{\mathcal{K}*}$:

$$0 \to \bar{g}_{\mathcal{K}*} j_\# \underline{\tilde{\mathcal{F}}}^{(n)} \to \bar{g}^\infty_{\mathcal{K}*} \underline{\mathcal{F}}_{\mathcal{K}}^{(n),\infty} \to \underline{\mathcal{S}}_n(\mathcal{K}) \to \underline{\mathcal{S}}_n^2(\mathcal{K}) \to 0, \tag{72}$$

where $\bar{g}^\infty_{\mathcal{K}} : \mathfrak{M}^\infty_{\mathcal{K}} \to \operatorname{Spec} A(\mathfrak{n})$ is the structure morphism constructed in Section 2.

Because all the modules in the short exact sequence (71) are flat crystals, so are the four modules in the above 4-term sequence, cf. [4], Ch. 6. By the argument given at the end of the above proof, Lemma 12.4 yields the following:

Corollary 12.5 *The crystals $\mathfrak{b}^*_{K_\infty} \underline{\mathcal{S}}_n^2(\mathcal{K})$ and $\mathfrak{b}^*_{K_\infty}(\mathrm{Ker}(\underline{\mathcal{S}}_n(\mathcal{K}) \to \underline{\mathcal{S}}_n^2(\mathcal{K})))$ are uniformizable in the sense of Definition 9.17.*

Let us now give an explicit representative of $\underline{\mathcal{H}}_{\mathcal{K},n} := \mathfrak{b}_K^*(\mathrm{Ker}(\underline{\mathcal{S}}_n(\mathcal{K}) \to \underline{\mathcal{S}}_n^2(\mathcal{K})))$. Define $\underline{\mathcal{H}}'_{\mathcal{K},n}$ as the τ-sheaf $H^0(\mathfrak{M}^\infty_{\mathcal{K},K}, \mathfrak{b}_K^* \underline{\mathcal{F}}_{\mathcal{K}}^{(n-2),\infty})$. Then the sheaf underlying $\underline{\mathcal{H}}_{\mathcal{K}',n}$ is projective over $K \otimes A$. Its rank is given by the number of cusps $h(\mathcal{K})$ (of $\mathfrak{M}_{\mathcal{K},K^{\mathrm{alg}}}$). More precisely, if for each cusp c of $\mathfrak{M}_{\mathcal{K},K}$ by K_c we denote the field of definition of c, so that K_c is a finite extension of K, then $\underline{\mathcal{H}}'_{\mathcal{K},n} = \bigoplus_c \mathbb{1}_{\operatorname{Spec} K_c, A} \otimes_A P_c^{n-2}$. Furthermore for each component c' of $\mathfrak{M}_{\mathcal{K},K}$ by $K'_{c'}$ we denote the constant field of this component c', so that $K'_{c'}$ is a finite extension of K. Then $H^0(\overline{\mathfrak{M}}_{\mathcal{K},K}, \mathbb{1}_{\overline{\mathfrak{M}}_{\mathcal{K},K},A}) = \bigoplus_{c'} \mathbb{1}_{\operatorname{Spec} K'_{c'},A}$, and so the latter module is of rank $d(\mathcal{K})$ over $K \otimes A$, where $d(\mathcal{K})$ is the number of components of $\mathfrak{M}_{\mathcal{K},K^{\mathrm{alg}}}$.

For $n > 2$ one has $s_n(\mathcal{K}) - s_n^2(\mathcal{K}) = h(\mathcal{K})$. By the previous corollary, $\mathfrak{b}^*_{K_\infty} \underline{\mathcal{H}}_{\mathcal{K},n}$ has a projective representative of rank $h(\mathcal{K})$ over $K_\infty \otimes A$ on which τ is injective. Regarded as crystals we have a surjection $\mathfrak{b}^*_{K_\infty} \underline{\mathcal{H}}'_{\mathcal{K},n} \twoheadrightarrow \mathfrak{b}^*_{K_\infty} \underline{\mathcal{H}}_{\mathcal{K},n}$. Considering the ranks, it must be an isomorphism, and this easily implies that $\underline{\mathcal{H}}_{\mathcal{K}',n}$ is a representative of $\underline{\mathcal{H}}_{\mathcal{K},n}$.

On the other hand, if $n = 2$, then one simply has $j_{\mathcal{K},\#}\underline{\mathcal{F}}_{\mathcal{K}}^{(0)} \cong \mathbb{1}_{\overline{\mathfrak{M}}_{\mathcal{K}},A}$, and a representative of $\underline{\mathcal{H}}_{\mathcal{K},n}$ is given by

$$\mathrm{Coker}(H^0(\overline{\mathfrak{M}}_{\mathcal{K},K}, \mathbb{1}_{\overline{\mathfrak{M}}_{\mathcal{K},K},A}) \hookrightarrow H^0(\mathfrak{M}^\infty_{\mathcal{K},K}, \mathbb{1}_{\mathfrak{M}^\infty_{\mathcal{K},K},A})).$$

We summarize the above in:

Corollary 12.6 *The kernel $\mathfrak{b}_K^*(\mathrm{Ker}(\underline{\mathcal{S}}_n(\mathcal{K}) \to \underline{\mathcal{S}}_n^2(\mathcal{K})))$ is represented by*

$$H^0(\mathfrak{M}^\infty_{\mathcal{K},K}, \mathfrak{b}_K^* \underline{\mathcal{F}}_{\mathcal{K}}^{(n-2),\infty})$$

if $n > 2$. If $n = 2$, it is represented by

$$\mathrm{Coker}\left(H^0(\overline{\mathfrak{M}}_{\mathcal{K},K}, \mathbb{1}_{\overline{\mathfrak{M}}_{\mathcal{K},K},A}) \longrightarrow H^0(\mathfrak{M}^\infty_{\mathcal{K},K}, \mathbb{1}_{\mathfrak{M}^\infty_{\mathcal{K},K},A})\right).$$

By Corollary 11.52 and Lemma 2.11 the τ-sheaf $\mathfrak{b}^*_{K_\infty}\underline{\mathcal{H}}_{\mathcal{K}',n}$ is isomorphic to

$$\bigoplus_{\tilde{s}=([\underline{s}],g\mathcal{K})\in \mathrm{GL}_2(K)\backslash(\mathbb{P}^1(K)\times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K})} \underline{\mathbb{1}}_{\mathrm{Spec}\, K_\infty, A} \otimes_A \mathrm{Hom}((\Lambda_g)_{\Gamma_{\tilde{s}}}, \Omega_A)^{\otimes(n-2)},$$

where $\Gamma_{\tilde{s}}$ is the stabilizer of the end $[\tilde{s}]$. Therefore taking τ-invariants yields:

Lemma 12.7 *There is an isomorphism*

$$(\underline{\mathcal{H}}^{\mathfrak{O}_A\text{-rig}}_{\mathcal{K}',n})^\tau \cong \bigoplus_{\tilde{s}\in \mathrm{GL}_2(K)\backslash(\mathbb{P}^1(K)\times \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K})} \mathrm{Hom}((\Lambda_g)_{\Gamma_{\tilde{s}}}, \Omega_A)^{\otimes(n-2)}$$

of projective A-modules of rank $h(\mathcal{K})$.

As a corollary to the above lemma and Theorems 10.3 and 12.3, we find:

Corollary 12.8 *As an A-module, the cokernel of $\mathbf{C}^{\mathrm{St},2}_n(\mathcal{K}, A) \hookrightarrow \mathbf{C}^{\mathrm{St}}_n(\mathcal{K}, A)$ is isomorphic to*

$$\bigoplus_{s\in\mathcal{K}\backslash \mathrm{GL}_2(\mathbb{A}^f)/K^* B(\mathbb{A}^f)} \Big((\Lambda_s)_{\Gamma_s} \otimes \Omega^*_A\Big)^{\otimes(n-2)}$$

for $n > 2$. For $n = 2$ it is isomorphic to $A^{h(\mathcal{K})-d(\mathcal{K})}$.

Remark 12.9 The fact that the cokernel of $\mathbf{S}^2_n(\mathcal{K}) \to \mathbf{S}_n(\mathcal{K})$ has such a natural presentation poses the question of whether one can give a simple basis for it, perhaps even in terms of Hecke eigenforms. Since Poincaré series can often be used to give a set of representatives of cusp forms modulo double cusp forms, perhaps they are also useful in constructing eigenforms which are not doubly cuspidal. We hope to come back to this in future work.

13 The Hecke-action on cohomology

13.1 Hecke-operators

For the definition of Hecke operators on cohomology, we follow the classical example of modular forms. We fix an admissible subgroup $\mathcal{K}$ of $\mathrm{GL}_2(\hat{A})$ and an element y in the corresponding subgroup $\mathcal{Y}$ of $\mathrm{GL}_2(\mathbb{A}^f)$. We define $\mathcal{K}' := y\mathcal{K}y^{-1} \cap \mathcal{K}$ and $\mathcal{K}'' := y^{-1}\mathcal{K}y \cap \mathcal{K}$. Let us consider the following diagram

$$\begin{tikzcd}
 & \pi_1^*\underline{\mathcal{F}}_{\mathcal{K}} \cong \underline{\mathcal{F}}_{\mathcal{K}'} \overset{\alpha_y}{\longleftarrow} \bar{\pi}_2^*\underline{\mathcal{F}}_{\mathcal{K}} & & \underline{\mathcal{F}}_{\mathcal{K}''} \cong \pi_2^*\underline{\mathcal{F}}_{\mathcal{K}} & \\
\underline{\mathcal{F}}_{\mathcal{K}} \arrow[d, dash, dashed] & \mathfrak{M}_{\mathcal{K}'} \arrow[u, dash, dashed] \arrow[rr, "r_y"] \arrow[dl, "\pi_1"'] \arrow[drrr, "\bar{\pi}_2"] & & \mathfrak{M}_{\mathcal{K}''} \arrow[u, dash, dashed] \arrow[dr, "\pi_2"] & \underline{\mathcal{F}}_{\mathcal{K}} \arrow[d, dash, dashed] \\
\mathfrak{M}_{\mathcal{K}} & & & & \mathfrak{M}_{\mathcal{K}},
\end{tikzcd} \tag{73}$$

which displays moduli schemes together with their universal Drinfeld modules (the relationship is indicated by dashed lines). The π_i are the natural projections that arise from the inclusions $\mathcal{K}', \mathcal{K}'' \subset \mathcal{K}$. The map r_y is induced from the morphism of Drinfeld moduli schemes given by right multiplication with y on level structures. Because $y \in M_2(\hat{A}) \cap \mathrm{GL}_2(\mathbb{A}^f)$, the map r_y is given by an isogeny. This isogeny induces the map $\alpha_y : \bar{\pi}_2^*\underline{\mathcal{F}}_{\mathcal{K}} \longrightarrow \underline{\mathcal{F}}_{\mathcal{K}'}$. By Theorem 2.14, the maps π_i are finite flat. Also the map r_y is finite flat. All morphisms are considered over $\operatorname{Spec} A(\mathfrak{n})$ where $\mathfrak{n}$ is the minimal conductor of $\mathcal{K}$. The map induced from r_y on the $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ is again denoted α_y.

We have the adjunction map $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} \to \pi_{1*}\pi_1^*\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ of Proposition 7.9. Because π_2 is finite flat and $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ is locally free, there is also a trace map $\pi_{2*}\pi_2^*\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} \to \underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ defined above Proposition 7.10. Recall that the structure morphism $\mathfrak{M}_{\mathcal{K}} \to \operatorname{Spec} A(\mathfrak{n})$ is denoted $g_{\mathcal{K}}$. By finiteness of $\bar{\pi}_2$ and π_1, the spectral sequence for the composite of pushforwards yields the isomorphisms

$$R^i g_{\mathcal{K}!}\pi_{1*}\underline{\mathcal{F}}_{\mathcal{K}'}^{(n)} \xrightarrow{\cong} R^i g_{\mathcal{K}'!}\underline{\mathcal{F}}_{\mathcal{K}'}^{(n)} \tag{74}$$

$$R^i g_{\mathcal{K}!}\bar{\pi}_{2*}\underline{\mathcal{F}}_{\mathcal{K}'}^{(n)} \xrightarrow{\cong} R^i g_{\mathcal{K}'!}\underline{\mathcal{F}}_{\mathcal{K}'}^{(n)}. \tag{75}$$

We now define the action of $\mathcal{K}y\mathcal{K}$ on $\underline{\mathcal{S}}_n(\mathcal{K}) = R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ as the following composite of morphisms:

$$\begin{aligned} R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} &\xrightarrow{\text{adj}} R^1 g_{\mathcal{K}!}\bar{\pi}_{2*}\bar{\pi}_2^*\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} \xrightarrow{(74)} R^1 g_{\mathcal{K}'!}\bar{\pi}_2^*\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} \xrightarrow{R^1\alpha_y} R^1 g_{\mathcal{K}'!}\pi_1^*\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} \\ &\xleftarrow{(75)} R^1 g_{\mathcal{K}!}\pi_{1*}\pi_1^*\underline{\mathcal{F}}_{\mathcal{K}}^{(n)} \xrightarrow{\text{trace}} R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}. \end{aligned}$$

Proposition 13.1 *The above definition of an action of $\mathcal{K}y\mathcal{K}$ on $R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ can be extended linearly to a right action of $\mathcal{H}(\mathcal{K}, \mathcal{Y})$.*

We leave the details of the proof to the reader, as it is a standard argument for geometric correspondences. We note that it is natural to have a right action on cohomology classes and a left action on forms, because Theorem 10.3 identifies the dual of the space of forms with the τ-invariants of the natural $\mathfrak{D}_A$-rigid sheaf on K_∞ attached to $R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$, and because it is well-known that under a duality a left action turns into a right action.

As the above construction is purely geometric, it can also be applied on the rigid site to the crystal $\underline{\mathcal{S}}_n^{\mathfrak{D}_A\text{-rig}}(\mathcal{K})$, and on the étale site to the étale A_v-sheaves

$\underline{\mathcal{S}}_n^{\text{ét},v}(\mathcal{K})$ for any place v of A, yielding a right action of $\mathcal{H}(\mathcal{K},\mathcal{Y})$. Clearly the functors $\underline{\mathcal{S}}_n(\mathcal{K}) \mapsto \underline{\mathcal{S}}_n(\mathcal{K})^{\mathfrak{D}_A\text{-rig}}$ and $\underline{\mathcal{S}}_n(\mathcal{K}) \mapsto \underline{\mathcal{S}}_n(\mathcal{K})^{\text{ét},v}$ are Hecke-equivariant.

The above construction also yields an operation of $\mathcal{H}(\mathcal{K},\mathcal{Y})$ on the crystal of Drinfeld double cusp forms. One has to replace $\underline{\mathcal{F}}_{\tilde{\mathcal{K}}}^{(n)}$ by its maximal extension to the cusps and $g_{\tilde{\mathcal{K}}}$ by $\bar{g}_{\tilde{\mathcal{K}}}$. It is therefore easy to see that the so-constructed Hecke operation is compatible with the surjection $\underline{\mathcal{S}}_n(\mathcal{K}) \twoheadrightarrow \underline{\mathcal{S}}_n^2(\mathcal{K})$.

13.2 Hecke-equivariance of the Eichler-Shimura isomorphism

We now prove the following central result on the Eichler-Shimura isomorphisms for Drinfeld cusp and double cusp forms:

Theorem 13.2 *For each admissible $\mathcal{K}$, the isomorphisms*

$$\Big(\mathbf{C}_n^{\text{St}}(\mathcal{K},A)\Big)^* \cong \Big(\underline{\mathcal{S}}_n^{\mathfrak{D}_A\text{-rig}}(\mathcal{K})\Big)^\tau \quad \text{and} \quad \Big(\mathbf{C}_n^{\text{St},2}(\mathcal{K},A)\Big)^* \cong \Big(\underline{\mathcal{S}}_n^{2,\mathfrak{D}_A\text{-rig}}(\mathcal{K})\Big)^\tau$$

of Theorems 10.3 and 12.3 are Hecke-equivariant.

We have not given a proof of Proposition 6.26, which asserts that the integral Steinberg cycles are stable under the Hecke action. Below we will prove that, viewed as a submodule of $\mathbf{C}_n^{\text{St}}(\mathcal{K},K)^*$, the Hecke action on $(\mathbf{C}_n^{\text{St}}(\mathcal{K},A))^*$ given in Definition 6.19 agrees via our Eichler-Shimura isomorphism with the cohomologically defined Hecke-action on $(\underline{\mathcal{S}}_n^{\mathfrak{D}_A\text{-rig}}(\mathcal{K}))^\tau$. Therefore this will imply that the Hecke action preserves $\mathbf{C}_n^{\text{St}}(\mathcal{K},A)$. So a posteriori, the proof below will also show Proposition 6.26.

Proof: Clearly it suffices to show Hecke-equivariance for the first isomorphism. The proof is based on an explicit computation using Čech-complexes for the standard affinoid covers $\mathfrak{U}_{\tilde{\mathcal{K}}}$ of the spaces $\overline{\mathfrak{M}}_{\tilde{\mathcal{K}}}^{\text{rig}}$ for $\tilde{\mathcal{K}} \in \{\mathcal{K},\mathcal{K}'\}$. We denote by $\pi_1^{-1}(\mathfrak{U}_{\mathcal{K}})$ and $\bar{\pi}_2^{-1}(\mathfrak{U}_{\mathcal{K}})$ the respective pullbacks of the affinoid cover $\mathfrak{U}_{\mathcal{K}}$. By finite flatness of π_1 and $\bar{\pi}_2$, the resulting covers are again affinoid covers of $\overline{\mathfrak{M}}_{\mathcal{K}'}^{\text{rig}}$.

The Čech complex $\mathcal{C}^\bullet(\mathfrak{U}_{\tilde{\mathcal{K}}}, \underline{\tilde{\mathcal{F}}}_{\tilde{\mathcal{K}}}^{(n)})$ has non-vanishing cohomology only in degree one, and there it is isomorphic to $H^1(\overline{\mathfrak{M}}_{\tilde{\mathcal{K}},K_\infty}^{\text{rig}}, \underline{\tilde{\mathcal{F}}}_{\tilde{\mathcal{K}}}^{(n)})$. Also the Čech complex is concentrated in degrees zero and one, and so one has short exact sequences with right hand term the H^1-term. In this situation, the proof of Theorem 10.3 shows that taking τ-invariants is an exact operation. Therefore to prove the desired Hecke-equivariance, we will consider the following diagram

$$\begin{tikzcd}
\mathcal{C}^0(\mathfrak{U}_{\mathcal{K}}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\tilde{\mathcal{K}}})^\tau \arrow[r, hook] \arrow[d, "\mathrm{adj}"] & \mathcal{C}^1(\mathfrak{U}_{\mathcal{K}}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[d, "\mathrm{adj}"] & (1) \\
\mathcal{C}^0(\mathfrak{U}_{\mathcal{K}}, \bar{\pi}_{2*}\bar{\pi}_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[r, hook] \arrow[d, "\cong"] & \mathcal{C}^1(\mathfrak{U}_{\mathcal{K}}, \bar{\pi}_{2*}\bar{\pi}_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[d, "\cong"] & (2) \\
\mathcal{C}^0(\bar{\pi}_2^{-1}(\mathfrak{U}_{\mathcal{K}}), \bar{\pi}_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\mathrm{quis}"] & \mathcal{C}^1(\bar{\pi}_2^{-1}(\mathfrak{U}_{\mathcal{K}}), \bar{\pi}_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[d] & (3) \\
\mathcal{C}^0(\mathfrak{U}_{\mathcal{K}'}, \bar{\pi}_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[r, hook] \arrow[d, "\alpha_y"] & \mathcal{C}^1(\mathfrak{U}_{\mathcal{K}'}, \bar{\pi}_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[d, "\alpha_y"] & (4) \\
\mathcal{C}^0(\mathfrak{U}_{\mathcal{K}'}, \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[r, hook] \arrow[dr, phantom, "\mathrm{quis}"] & \mathcal{C}^1(\mathfrak{U}_{\mathcal{K}'}, \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau & (5) \\
\mathcal{C}^0(\pi_1^{-1}(\mathfrak{U}_{\mathcal{K}}), \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[r, hook] \arrow[u] \arrow[d, "\cong"] & \mathcal{C}^1(\pi_1^{-1}(\mathfrak{U}_{\mathcal{K}}), \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[u] \arrow[d, "\cong"] & (6) \\
\mathcal{C}^0(\mathfrak{U}_{\mathcal{K}}, \pi_{1*}\pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[r, hook] \arrow[d, "\mathrm{trace}"] & \mathcal{C}^1(\mathfrak{U}_{\mathcal{K}}, \pi_{1*}\pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[d, "\mathrm{trace}"] & (7) \\
\mathcal{C}^0(\mathfrak{U}_{\mathcal{K}}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \arrow[r, hook] & \mathcal{C}^1(\mathfrak{U}_{\mathcal{K}}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau, & (8)
\end{tikzcd}$$

whose cokernel sequence is given by

$$H^1(\mathfrak{U}_{\mathcal{K}}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \xrightarrow{\mathrm{adj}} H^1(\mathfrak{U}_{\mathcal{K}}, \pi_{2*}\pi_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \longrightarrow H^1(\bar{\pi}_2^{-1}(\mathfrak{U}_{\mathcal{K}}), \bar{\pi}_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau$$
$$\xrightarrow{\cong} H^1(\mathfrak{U}_{\mathcal{K}'}, \bar{\pi}_2^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \xrightarrow{\alpha_y} H^1(\mathfrak{U}_{\mathcal{K}'}, \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \xleftarrow{\cong} H^1(\pi_1^{-1}(\mathfrak{U}_{\mathcal{K}}), \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau$$
$$\xrightarrow{\cong} H^1(\mathfrak{U}_{\mathcal{K}}, \pi_{1*}\pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \xrightarrow{\mathrm{trace}} H^1(\mathfrak{U}_{\mathcal{K}}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau,$$

and is the defining sequence of the Hecke operator $_{|\mathcal{K}y\mathcal{K}}$. By the isomorphisms displayed in (74) and (75), the map of complexes from line (2) to line (3) and from line (6) to line (5) are quasi-isomorphisms. The corresponding maps on the level of Čech complexes are the maps induced from refining the Čech covers $\bar{\pi}_2^{-1}(\mathfrak{U}_{\mathcal{K}})$, respectively $\pi_1^{-1}(\mathfrak{U}_{\mathcal{K}})$ to $\mathfrak{U}_{\mathcal{K}'}$. All horizontal maps are boundary maps.

We now want to compute the Hecke-operation for elements of the cokernel of the first line. This can be done by computing the image of an element $f_1 \in \mathcal{C}^1(\mathfrak{U}_{\mathcal{K}}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau$ under the composite of all vertical maps. Let f_i be the image in line (i), $i = 1, \ldots, 8$, where given f_5, we choose an element f_6 such that the image of f_6 under

$$\mathcal{C}^1(\pi_1^{-1}(\mathfrak{U}_{\mathcal{K}}), \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \longrightarrow \mathcal{C}^1(\mathfrak{U}_{\mathcal{K}'}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}'})^\tau$$

is cohomologous to f_5.

To have a good combinatorial description of the covers $\mathfrak{U}_{\tilde{\mathcal{K}}}$, we use the same convention as in the proof of Theorem 10.3:

(i) For the cuspidal affinoids, we make any choices.

(ii) For affinoids corresponding to stable simplices, we choose representatives for the the stable vertices, edges and oriented edges. The affinoids are denoted $\mathfrak{U}_t$.

(iii) We use the covering map $\Omega_{\mathcal{K}} \twoheadrightarrow (\mathfrak{b}^*_{K_\infty}\mathfrak{M}_{\mathcal{K}})^{\mathrm{rig}}$ and the choices in (ii) to identify the affinoids $\mathfrak{U}_{\tilde{t}}$ of $(\mathfrak{b}^*_{K_\infty}\mathfrak{M}_{\mathcal{K}})^{\mathrm{rig}}$ with the affinoids $\mathfrak{U}'_{\tilde{t}}$ of $\Omega_{\mathcal{K}}$.

(iv) We have an action of $\mathrm{GL}_2(K)$ on the standard affinoids $\mathfrak{U}_{\tilde{t}}$ of $\Omega_{\mathcal{K}}$ and their intersections.

We may evaluate f on all affinoids $\mathfrak{U}'_{\tilde{t}}$ with $\tilde{t}$ stable and on their intersections. Therefore there is no need to explicitly give names to the representatives chosen in (i) and (ii). In particular for any $\tilde{e} = (e, g\mathcal{K}) \in \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}$, we have a value

$$f_1(\tilde{e}) \in \Gamma(\mathfrak{U}'_{\tilde{e}}, \mathrm{Sym}^n \underline{\tilde{\mathcal{F}}}(\mathcal{K}))^\tau \cong \mathrm{Sym}^n \mathrm{Hom}_A(\Lambda_g, \Omega_A) =: M_g.$$

Also, it is quite easy to combinatorially describe the maps induced by $\bar{\pi}_2$ and π_1 for the spaces $\Omega_{\tilde{\mathcal{K}}}$.

A simple computation yields

$$f_5(\tilde{e}') = \begin{cases} f_1(\bar{\pi}_2(\tilde{e}')) & \text{if } \bar{\pi}_2(\tilde{e}') \text{ is stable} \\ 0 & \text{otherwise.} \end{cases}$$

The only non-trivial step in computing f_8 is to give an algorithm which computes f_6 from f_5. Because the map from line (6) to line (5) is a quasi-isomorphism, we know that the element f_5 can be altered by adding a suitable coboundary f_5^0 coming from $\mathcal{C}^0(\mathfrak{U}_{\mathcal{K}'}, \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau$ in such a way that $f'_5 := f_5 + f_5^0$ lies in the image of that map. To be in the image of $\mathcal{C}^1(\pi_1^{-1}(\mathfrak{U}_{\mathcal{K}}), \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau$ means that f'_5 takes the value zero on all oriented stable edges $\tilde{e}'$ such that $\pi_1(\tilde{e}')$ is unstable. Thus we need to find f_5^0 such that f'_5 takes the value zero on such $\tilde{e}'$.

As in the construction of Hecke operators on Steinberg cycles, the concept of sources will play the essential role.

Definition 13.3 *For an edge $\tilde{e}'$ of $\mathcal{T}^o_{\mathcal{K}',1}$, one defines its π_1-source $\mathrm{src}_{\pi_1}(\tilde{e}') \subset \mathcal{T}^o_{\mathcal{K}',1}$ as*

$\{\underline{\tilde{e}}' \in \mathcal{T}^o_{\mathcal{K}',1} : \pi_1(\underline{\tilde{e}}') \in \mathrm{src}(\pi_1(\tilde{e}'))$ *and $\underline{\tilde{e}}'$ is in the same component of $\mathcal{T}_{\mathcal{K}'}$ as $\tilde{e}'$*$\}$.

Note that $\pi_1^{-1}(\mathrm{src}(\pi_1(\tilde{e}')))$ is automatically contained in $\mathcal{T}^{o,\mathrm{st}}_{\mathcal{K}',1}$, so that in fact $\mathrm{src}_{\pi_1}(\tilde{e}') \subset \mathcal{T}^{o,\mathrm{st}}_{\mathcal{K}',1}$. Using the finiteness of π_1, it is easy to see that $\mathrm{src}_{\pi_1}(\tilde{e}')$ is finite for any $\tilde{e}' \in \mathcal{T}^o_{\mathcal{K}',1}$.

Claim: If $\tilde{e}'$ is stable, then the edges in $\mathrm{src}_{\pi_1}(\tilde{e}')$ together with $\tilde{e}'$ form the boundary of a finite connected subgraph $S_{\tilde{e}'}$ in the stable region of $\mathcal{T}_{\mathcal{K}'}$:
The claim is obvious if $\pi_1(\tilde{e}')$ is stable, and so we assume otherwise. For $\underline{\tilde{e}}' \in \mathrm{src}_{\pi_1}(\tilde{e}')$ we denote by $\underline{s}$ be the rational half line through $\tilde{e}'$ which starts at $\underline{\tilde{e}}'$ and whose image $\underline{s}'$ under π_1 is in the unstable part of $\mathcal{T}_{\mathcal{K}}$ except for $\pi(\underline{\tilde{e}}')$. While moving along $\underline{s}'$ from $\pi_1(\underline{\tilde{e}}')$ to $\pi_1(\tilde{e}')$, the stabilizers of the respective edges do not become smaller. Therefore the stabilizers of the edges between $\underline{\tilde{e}}'$ and $\tilde{e}'$ exhibit the same behavior, and hence they are all trivial. The claim is thus shown.

Let $t = \pi_1(\tilde{e})$, if $\pi_1(\tilde{e})$ is stable, and otherwise let t be the cusp corresponding to the end of $\mathrm{GL}_2(K)\backslash\mathcal{T}_{\mathcal{K}}$ which contains $\pi_1(\tilde{e})$. Then the affinoids $\mathfrak{U}_{\underline{\tilde{e}}'}$ for $\underline{\tilde{e}}' \in S_{\tilde{e}'}$ are in one connected component of the affinoid $\pi_1^{-1}(\mathfrak{U}_t)$. Furthermore for all stable simplices $\tilde{t}$ of $\mathcal{T}_{\mathcal{K}}$ one has

$$\Gamma(\pi_1^{-1}(\mathfrak{U}'_{\tilde{t}}), \pi_1^*\underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}})^\tau \cong \bigoplus_{\tilde{t}' \in \pi_1^{-1}(\tilde{t})} \Gamma(\mathfrak{U}'_{\tilde{t}'}, \underline{\tilde{\mathcal{F}}}^{(n)}_{\mathcal{K}'})^\tau.$$

Let f_5'' be the 1-cocycle which takes the value $f_5(\mathfrak{U}_{\tilde{e}'})$ on $\tilde{e}'$, the value $-f_5(\mathfrak{U}'_{\tilde{e}'})$ on $\mathrm{src}_{\pi_1}(\tilde{e}')$ and zero on the remaining oriented edges.

Claim: The 1-cocycle f_5'' is a coboundary.

Note that if the claim is shown, it provides an algorithm to define a coboundary f_5^0 such that $f_5' := f_5 + f_5^0$ arises from a function f_6. The proof of the claim is a simple combinatorial exercise. Let us induct on the size of the finite graph $S_{\tilde{e}'}$. Observe first, that by using 0-cocycles supported on non-oriented edges, we may always assume that $\pi_1(\tilde{e}')$ points toward an end (we use this 0-cocycle to change the orientation of all boundary edges of the graph). In this case, we first use a 0-cocycle with support on the non-oriented edge $\overline{\tilde{e}'}$ to obtain the value 0 on $\tilde{e}'$ and $f_6(\mathfrak{U}'_{\tilde{e}'})$ on $-\tilde{e}'$ and then we use the unique vertex $\tilde{v}'$ which lies at the tip of $-\tilde{e}'$ to obtain the value 0 on $-\tilde{e}'$ as well and the value $f_6(\tilde{e}')$ on all other adjacent edges of $\tilde{v}''$ which are not in $\mathrm{src}_{\pi_1}(\tilde{e}')$ and the value 0 on the other adjacent edges. All these edges are contained in the graph $S_{\tilde{e}'}$. If we remove $\tilde{v}'$ and $\overline{\tilde{e}'}$ from the graph, it decomposes into at most $|k|$ disjoint connected subgraphs, to all of which the inductive hypothesis applies. Thus the claim is shown.

Next we give an expression for f_8 in terms of f_1. The trace map, as well as the map from line (6) to line (7) are easy to compute on the level of the Čech complex. Using the above algorithm, one obtains the following explicit formula:

$$\begin{aligned} f_8(\tilde{e}) &= \sum_{\tilde{e}' \in \pi_1^{-1}(\tilde{e})} f_6(\tilde{e}') \\ &= \sum_{\tilde{e}' \in \pi_1^{-1}(\tilde{e})} \sum_{\substack{\tilde{e}' \in \mathrm{src}_{\pi_1}(\underline{\tilde{e}}') \\ \underline{\tilde{e}}' \in \mathcal{T}^{\mathrm{st},o}_{\mathcal{K}',1}}} f_5(\underline{\tilde{e}}') \\ &= \sum_{\tilde{e}' \in \pi_1^{-1}(\tilde{e})} \sum_{\substack{\tilde{e}' \in \mathrm{src}_{\pi_1}(\underline{\tilde{e}}') \\ \bar{\pi}_2(\underline{\tilde{e}}') \in \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}}} f_1(\bar{\pi}_2(\underline{\tilde{e}}')). \end{aligned}$$

We now use the following notation: Say $\tilde{e} = (e, h\mathcal{K})$, $\underline{\tilde{e}'} = (e', g\mathcal{K}')$. Say we have $\mathcal{K} = \amalg z_j \mathcal{K}'$. Then $\mathcal{K}y\mathcal{K} = \amalg y_j \mathcal{K}$ with $y_j = z_j y$. Summing over the $\tilde{e}'$ in $\pi_1^{-1}(\tilde{e})$ is the same as summing over the $(e, hz_j\mathcal{K}')$. Thus we have

$$\begin{aligned} f_8(\tilde{e}) &= \sum_j \sum_{\substack{(e,hz_j\mathcal{K}') \in \mathrm{src}_{\pi_1}(e',g\mathcal{K}') \\ (e',gy\mathcal{K}) \in \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}}} f_1((e', gy\mathcal{K})) \\ &= \sum_j \sum_{\substack{(e,h\mathcal{K}) \in \mathrm{src}(e',h\mathcal{K}) \\ (e',hy_j\mathcal{K}) \in \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}}} f_1((e', hy_j\mathcal{K})) \\ &= \sum_j \sum_{\substack{\tilde{e} \in \mathrm{src}(e',\tilde{g}y_j^{-1}\mathcal{K}) \\ (e',\tilde{g}\mathcal{K}) \in \mathcal{T}^{\mathrm{st},o}_{\mathcal{K},1}}} f_1((e', \tilde{g}\mathcal{K})). \end{aligned}$$

As in the proof of Theorem 10.3, let $\bar{M}$ be the local system defined by the M_g. Our conventions regarding the covers $\mathfrak{U}_t$ and $\mathfrak{U}'_t$ identify f_1 with an element in $\mathrm{Hom}_{\mathrm{GL}_2(k)}(\bar{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},1}, \bar{M})$. The computation just performed shows that under this identification the cohomologically defined Hecke operation of $\mathcal{K}y\mathcal{K}$ agrees, up to coboundaries, with $f_1 \circ \_\_|\mathcal{K}y\mathcal{K}$, where $\_\_\mathcal{K}y\mathcal{K}$ is the Hecke action on $\bar{\mathcal{C}}^{\mathrm{st}}_{\mathcal{K},1}$ of Definition 6.19. But the latter is precisely the Hecke operation on $(\mathbf{C}^{\mathrm{St}}_n(\mathcal{K}, A))^*$, and thus we have completed the proof of Theorem 13.2. ∎

13.3 Decomposing the crystal $\underline{\mathcal{S}}_n(\mathcal{K})$

In this subsection we will show that $\underline{\mathcal{S}}_n(\mathcal{K})$ is representable by a torsion free τ-sheaf with injective τ. Once this has been accomplished, we define a (non-canonical) filtration on the representing τ-sheaf into saturated irreducible Hecke submodules. At the end, these subquotients are shown to be uniformizable crystals.

Proposition 13.4 *For every $n \in \mathbb{N}_0$ and any admissible $\mathcal{K}$ with minimal conductor $\mathfrak{n}$, the crystal $\underline{\mathcal{S}}_n(\mathcal{K})$ is representable by a τ-sheaf whose underlying module is torsion free on $A(\mathfrak{n}) \otimes A$ and on which τ is injective with cokernel of codimension at least one.*

PROOF: The scheme $\mathfrak{M}_{\mathcal{K}}$ is smooth and affine over $\operatorname{Spec} A(\mathfrak{n})$ of relative dimension one. Therefore by adding a τ-sheaf whose underlying module is projective and on which $\tau = 0$, we may represent the crystal $\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ by a free τ-sheaf $\underline{\mathcal{G}}$. In particular $\mathcal{G} \cong \mathcal{G}_0 \otimes_k A$ for some free sheaf $\mathcal{G}_0 \cong \mathcal{O}^l_{\mathfrak{M}_{\mathcal{K}}}$ on $\mathfrak{M}_{\mathcal{K}}$. By the remark after Theorem 7.12, there exists an $m > 0$ such that $j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}^{(n)}$ is representable by a τ-sheaf $\underline{\mathcal{G}}'$ whose underlying sheaf is $\mathcal{G}'_0 \otimes_k A$ with $\mathcal{G}'_0 \cong \mathcal{O}^l_{\overline{\mathfrak{M}}_{\mathcal{K}}}(m\mathfrak{M}_{\mathcal{K}}^{\infty})$. Because $\mathfrak{M}_{\mathcal{K}}^{\infty}$ is finite flat over $\operatorname{Spec} A(\mathfrak{n})$, it follows that $R^1_*\bar{g}_{\mathcal{K}}\mathcal{G}'_0$ has constant rank over $\operatorname{Spec} A(\mathfrak{n})$, i.e., that

$$\mathcal{H} := R^1_*\bar{g}_{\mathcal{K}}\mathcal{G}'_0 \otimes_k A \cong R^1_*\bar{g}_{\mathcal{K}}\mathcal{G}'$$

is locally free over $\operatorname{Spec} A(\mathfrak{n}) \times A$.

We have seen in Lemma 10.8 that there is an $\mathbb{N}$ such that $\mathfrak{b}_K^*\tau^j(\underline{\mathcal{H}})$ has the properties asserted for τ. Because $\tau^j(\mathcal{H})$ is a submodule of the free sheaf $\mathcal{H}$, it is torsion free. Furthermore generic injectivity of τ on $\tau^j(\underline{\mathcal{H}})$ implies injectivity of τ. Therefore $(\tau^j(\mathcal{H}), \tau)$ has all the properties asserted in the proposition. ∎

In the sequel we will write $\underline{\mathcal{S}}_n(\mathcal{K})$ for a τ-sheaf constructed in the proof of the proposition.

Remark 13.5 We would like to strengthen the above proposition by showing that for admissible $\mathcal{K}$ the crystal $\underline{\mathcal{S}}^2_n(\mathcal{K})$ is representable by a τ-sheaf which is locally free and on which τ is a monomorphism. As such a representation clearly exists for the kernel of $\underline{\mathcal{S}}_n(\mathcal{K}) \to \underline{\mathcal{S}}^2_n(\mathcal{K})$, this would give a decomposition of the crystal $\underline{\mathcal{S}}_n(\mathcal{K})$ into pieces which behave well under base and coefficient change on the level of τ-sheaves. Then one could often work in the more explicit category of τ-sheaves instead of the category of crystals. In the example of Section 15 the crystal $\underline{\mathcal{S}}^2_n(\mathcal{K})$ has indeed a locally free representative of the expected rank, but we have no general results.

With finite coefficients, the situation is much better understood. Namely, let $\mathfrak{p}, \mathfrak{q}$ be in $\operatorname{Max}(A)$ and denote by $\mathfrak{b}_{k_{\mathfrak{q}}^{\mathrm{alg}}}$ is the base change morphism corresponding to $A(\mathfrak{n}) \to k_{\mathfrak{q}}^{\mathrm{alg}}$ for a space above $\operatorname{Spec} A(\mathfrak{n})$. Then the results of Pink in [40] imply that $R^1\bar{g}_{\mathcal{K}*}j_{\mathcal{K}\#}\mathfrak{b}^*_{k_{\mathfrak{q}}^{\mathrm{alg}}} \operatorname{Sym}^n(\underline{\mathcal{F}}_{\mathcal{K}}/\mathfrak{p}\underline{\mathcal{F}}_{\mathcal{K}})$ can be represented by a locally free τ-sheaf of the expected rank on $\operatorname{Spec} k_{\mathfrak{p}}^{\mathrm{alg}}$ over $A/\mathfrak{p}$.

Let $f_1, \ldots, f_n$ be a $\mathbb{C}_\infty$-basis of $\mathbf{S}_n(\mathcal{K})$ such that for each j, the element f_{j+1} considered in $\mathbf{S}_n(\mathcal{K})/(f_1, \ldots, f_j)$ is a Hecke eigenform. Say we order the f_i such that we first have double cusp forms and then other forms. Furthermore, we order them so that forms conjugate under the action of $\operatorname{Gal}(K^{\mathrm{sep}}/K)$ are numbered consecutively. Note that given some f_j with Hecke eigenvalues a_h for $h \in \mathcal{H}(\mathcal{K}, \mathcal{Y})$, there exists a Hecke eigenform in $\mathbf{S}_n(\mathcal{K})$ with the same set of eigenvalues.

Let now $M'_0 \subset M'_1 \subset \ldots M'_k$ be the corresponding filtration of $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)^* \otimes_A K$, so that the successive subquotients correspond bijectively to the Galois conjugacy classes of the f_j, and each such subquotient is an irreducible $\mathcal{H}(\mathcal{K}, \mathcal{Y})$-module. Define the filtration M_i of $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)^* \cong \mathfrak{b}^*_{K_\infty} \underline{\mathcal{S}}_n(\mathcal{K})$ by intersecting the M'_i with $\mathbf{C}_n^{\mathrm{St}}(\mathcal{K}, A)^*$. The successive subquotients are projective over A and carry an action of $\mathcal{H}(\mathcal{K}, \mathcal{Y})$, i.e., there are maps $\mathcal{H}(\mathcal{K}, \mathcal{Y}) \to \mathrm{End}(M_i/M_{i-1})$. The image is denoted by A_i. It is a finite extension of A of degree equal to the rank of M_i/M_{i-1}. After tensoring with K over A, one has a non-canonical isomorphism $A_i \otimes_A K \cong (M_i/M_{i-1}) \otimes_A K$.

Let $\underline{\tilde{\mathcal{M}}}_i$ be the corresponding filtration on $(\mathfrak{b}^*_{K_\infty} \underline{\mathcal{S}}_n(\mathcal{K}))^{\mathfrak{D}_A\text{-rig}}$, so that $\underline{\tilde{\mathcal{M}}}_i \cong \tilde{\underline{\mathbb{1}}}_{\mathrm{Spm}\, K_\infty, \mathfrak{D}_A} \otimes_A M_i$. Because τ commutes with the Hecke action, the $\underline{\tilde{\mathcal{M}}}_i$ are Hecke-modules and on the subquotients the Hecke algebra acts in the obvious way by multiplication with A_i on M_i. Clearly $\mathfrak{b}^*_K \underline{\mathcal{S}}_n(\mathcal{K})$ is a sub-τ sheaf of $(\mathfrak{b}^*_{K_\infty} \underline{\mathcal{S}}_n(\mathcal{K}))^{\mathfrak{D}_A\text{-rig}}$, and its induced filtration is denoted by $\underline{\mathcal{M}}_i$. Comparing ranks, it follows that $\mathrm{rank}_A M_i = \mathrm{rank}_{K \otimes A} \underline{\mathcal{M}}_i$.

Definition 13.6 *Define $\underline{\mathcal{S}}_{n,i}(\mathcal{K})$ as the sub τ-sheaf of $\underline{\mathcal{S}}_n(\mathcal{K})$ obtained by intersecting $\underline{\mathcal{M}}_i$ with the sheaf underlying $\underline{\mathcal{S}}_n(\mathcal{K})$ and*

$$\overline{\underline{\mathcal{S}}}_{n,i} := \underline{\mathcal{S}}_{n,i}(\mathcal{K}) / \underline{\mathcal{S}}_{n,i-1}(\mathcal{K}).$$

In this way, we have attached to each conjugacy class of f_j a torsion free subquotient $\overline{\underline{\mathcal{S}}}_{n,i}(\mathcal{K})$ with injective τ-action, multiplication by A_i and generic rank equal to the number of conjugates of f_j, which in turn is equal to the rank of A_i over A. If the need arises, we write i_j for the i corresponding to j.

The assignment $f_j \mapsto \overline{\underline{\mathcal{S}}}_{n,i_j}(\mathcal{K})$ is in no way canonical. So in all the constructions below the independence of the chosen filtration will be an issue. However note that by the Jordan Hölder theorem the subquotients M'_i/M'_{i+1} are independent of the chosen filtration. Analogously, the subquotients $\underline{\mathcal{M}}_i/\underline{\mathcal{M}}_{i-1}$ if tensored with $\mathrm{Frac}(K \otimes A)$ over $K \otimes A$ are independent of any choices.

Note also that for any of the objects indexed by i, the action of $\mathcal{H}(\mathcal{K}, \mathcal{Y})$ is an action on coefficients in the following sense. Let $\tilde{A}$ be a finite extension of A which contains all the A_i. Then after change of coefficients $\otimes_A \tilde{A}$ and suitable identifications, the action of A_i is given by multiplication on the coefficients.

The above discussion, Corollary 12.5 and an inductive argument based on Lemma 12.4 show:

Proposition 13.7 *The crystals $\mathfrak{b}^*_{K_\infty} \overline{\underline{\mathcal{S}}}_{n,i}$ are uniformizable and can be represented by τ-sheaves which are projective of rank one over $K_\infty \otimes A_i$.*

Remark 13.8 In Section 15 we will compute an explicit example. It will turn out that the pieces $K_\infty \otimes A_i$ are pure, in a suitable sense. It seems likely to us that this holds in general, and we hope to come back to this question in future work.

13.4 The Eichler-Shimura relation

As in the classical case, there is an Eichler-Shimura relation for the geometric correspondence $T_\mathfrak{p}$ if we base change to the fiber at $\mathfrak{p}$. It gives an expression for $T_\mathfrak{p}$ in terms of the operation of Frobenius at $\mathfrak{p}$. This will eventually yield a characterization of Galois representations attached to cuspidal Drinfeld eigenforms.

We fix an admissible $\mathcal{K}$ with minimal conductor $\mathfrak{n}$ and $\mathfrak{p} \in \mathrm{Max}(A(\mathfrak{n}))$. Let $g_{\mathcal{K},\mathfrak{p}}$ denote the structure morphism of $\mathfrak{M}_{\mathcal{K},\mathfrak{p}}$ and $\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} := \mathfrak{b}^*_{k_\mathfrak{p}} \underline{\mathcal{F}}^{(n)}_{\mathcal{K}}$. Recall that the Hecke operator $T_{\mathfrak{p}}$ comes from the correspondence given by $y = \begin{pmatrix} \pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(A_{\mathfrak{p}})$, where $\pi_{\mathfrak{p}}$ is a uniformizer at $\mathfrak{p}$.

We write $F_{\mathfrak{p}}$ for the absolute Frobenius on $\mathfrak{M}_{\mathcal{K},k_{\mathfrak{p}}}$, i.e. for $\sigma^{d_{\mathfrak{p}}}_{\mathfrak{M}_{\mathcal{K},k_{\mathfrak{p}}}}$, where $d_{\mathfrak{p}} = [k_{\mathfrak{p}} : k]$. It induces an action on Drinfeld modules and level structures denoted

$$(\varphi, [\psi]) \overset{F_{\mathfrak{p}}}{\mapsto} (\varphi^{(\mathfrak{p})}, [\psi^{(\mathfrak{p})}]).$$

Since $\varphi \to \varphi^{(\mathfrak{p})}$ is a $\mathfrak{p}$-isogeny, the map $F_{\mathfrak{p}}$ on Drinfeld-modules fits into a diagram

$$\begin{tikzcd}
(\varphi, [\psi]) \arrow[r, mapsto, "F_{\mathfrak{p}}"'] \arrow[rr, mapsto, bend left, "I_{\mathfrak{p}}"] & (\varphi^{(\mathfrak{p})}, [\psi^{(\mathfrak{p})}]) \arrow[r, mapsto, "V_{\mathfrak{p}}"'] & (\varphi/\varphi[\mathfrak{p}], [\psi/\varphi[\mathfrak{p}]]),
\end{tikzcd}$$

where $I_{\mathfrak{p}}$ assigns to any pair $(\varphi, [\psi])$ its quotient by the full $\mathfrak{p}$-torsion scheme $\varphi[\mathfrak{p}]$.

Lemma 13.9 *Let $\alpha'_y \colon F^*_{\mathfrak{p}}\underline{\mathcal{M}}(\varphi) \cong \underline{\mathcal{M}}(F^*_{\mathfrak{p}}\varphi) \to \underline{\mathcal{M}}(\varphi)$ denote the map induced from the isogeny $F_{\mathfrak{p}} : \varphi \to F^*_{\mathfrak{p}}\varphi$, where $\underline{\mathcal{M}}(\varphi)$ is the τ-sheaf attached to the Drinfeld-module φ. Then one has*

$$\alpha'_y = \tau^{d_{\mathfrak{p}}}_{\mathcal{M}(\varphi)}$$

Proof: Let us explicitly describe the map $F_{\mathfrak{p}}$ on a Drinfeld-module

$$\varphi \colon A \to R\{\tau\} : a \mapsto \sum_i \alpha_i(a)\tau^i$$

in standard form. The Drinfeld-module $\varphi^{(\mathfrak{p})}$ is then given by

$$\varphi^{(\mathfrak{p})} \colon A \to R\{\tau\} : a \mapsto \sum_i \alpha_i^{q^{d_{\mathfrak{p}}}}(a)\tau^i$$

and the isogeny $\varphi \mapsto \varphi^{(\mathfrak{p})}$ is the element $\tau^{d_{\mathfrak{p}}} \in R\{\tau\}$, because one has

$$\tau^{d_{\mathfrak{p}}}\varphi = \varphi^{(\mathfrak{p})}\tau^{d_{\mathfrak{p}}}.$$

In this case $\underline{\mathcal{M}}(\varphi)$ is given as $(R\{\tau\}, \tau')$, where A acts on $R\{\tau\}$ by right composition with φ. The map $R\{\tau\} \to (\sigma \times \mathrm{id})_* R\{\tau\}$ in Example 7.7 (b) is given by left multiplication by τ. It easily follows that $\tau' \colon (\sigma \times \mathrm{id})^* R\{\tau\} \to R\{\tau\}$ is given by right multiplication with τ.

Let $E(\varphi)$ denote the A-module attached to φ. The isogeny $\tau^{d_{\mathfrak{p}}}$ gives rise to an element in $\mathrm{Hom}_{\mathfrak{G}/\operatorname{Spec} R}(\underline{\mathcal{M}}(\varphi), \underline{\mathcal{M}}(F^*_{\mathfrak{p}}\varphi))$, simply by composing with $\tau^{d_{\mathfrak{p}}} \in R\{\tau\} \cong \mathcal{M}(\varphi)$ on the right. It follows that $\tau^{d_{\mathfrak{p}}}$ operates in the same way as a morphism

$$\begin{aligned} F^*_{\mathfrak{p}}\underline{\mathcal{M}}(\varphi) &\cong \mathrm{Hom}_{\mathfrak{G}/\operatorname{Spec} R}(E(F^*_{\mathfrak{p}}\varphi), \mathbb{G}_a) \cong R\{\tau\} \\ \longrightarrow \quad \underline{\mathcal{M}} &\cong \mathrm{Hom}_{\mathfrak{G}/\operatorname{Spec} R}(E(\varphi), \mathbb{G}_a) \cong R\{\tau\}. \end{aligned}$$

Note that the A-operation on the two τ-sheaves displayed is different. By the previous paragraph, right multiplication with $\tau^{d_{\mathfrak{p}}}$ is simply the $d_{\mathfrak{p}}$-th iterate of τ'. Thus we have identified α'_y with ${\tau'}^{d_{\mathfrak{p}}}$. ∎

Theorem 13.10 (Eichler-Shimura relation) *The action of $T_{\mathfrak{p}}$ on the crystal $\mathfrak{b}^*_{k_{\mathfrak{p}}}\underline{\mathcal{S}}_n(\mathcal{K}) \cong R^1 g_{\mathcal{K},\mathfrak{p}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}}$ is given by*

$$R^1 g_{\mathcal{K},\mathfrak{p}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} = \sigma^*_{k_{\mathfrak{p}}} R^1 g_{\mathcal{K},\mathfrak{p}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \overset{\text{base change}}{\longrightarrow} R^1 g_{\mathcal{K},\mathfrak{p}!} F^*_{\mathfrak{p}}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \xrightarrow{\alpha'_y} R^1 g_{\mathcal{K},\mathfrak{p}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}}.$$

Proof: Define $\mathcal{K}' \subset \mathcal{K}$ such that $\mathfrak{M}_{\mathcal{K}'}$ classifies triples $(\varphi, [\psi], H)$ where φ is a Drinfeld-module, $[\psi]$ is a level $\mathcal{K}$-structure and H is a cyclic non-trivial $A/\mathfrak{p}$-submodule (scheme) of $\varphi[\mathfrak{p}]$. In terms of moduli, we have $\bar{\pi}_2 : (\varphi, [\psi], H) \mapsto (\varphi/H, [\psi/H])$ and $\pi_1 : (\varphi, [\psi], H) \mapsto (\varphi, [\psi])$ for the morphisms displayed in diagram (73). In analogy with the classical situation for elliptic curves one considers the following diagram

$$\begin{tikzcd}
 & \mathfrak{M}_{\mathcal{K},k_{\mathfrak{p}}} \amalg \mathfrak{M}_{\mathcal{K},k_{\mathfrak{p}}} \arrow[dl, "(\mathrm{id},F_{\mathfrak{p}})"'] \arrow[d, "\Phi=(\Phi_1,\Phi_2)"] \arrow[dr, "(F_{\mathfrak{p}},I_{\mathfrak{p}})"] & \\
\mathfrak{M}_{\mathcal{K},k_{\mathfrak{p}}} & \mathfrak{M}_{\mathcal{K}',k_{\mathfrak{p}}} \arrow[l, "\pi_1"] \arrow[r, "\bar{\pi}_2"'] & \mathfrak{M}_{\mathcal{K},k_{\mathfrak{p}}},
\end{tikzcd}$$

where the maps Φ_i are given in terms of moduli as

$$\Phi_1 : (\varphi, [\psi]) \mapsto (\varphi, [\psi], \operatorname{Ker} F_{\mathfrak{p}}) \quad \text{and} \quad \Phi_2 : (\varphi, [\psi]) \mapsto (\varphi^{(\mathfrak{p})}, [\psi^{(\mathfrak{p})}], \operatorname{Ker} V_{\mathfrak{p}}).$$

The maps are clearly well-defined, and, using a degree argument, one can show that away from the supersingular points the map Φ is an isomorphism. Pulling back the morphism $\alpha_y : \pi_2^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \to \bar{\pi}_1^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}}$ yields a direct sum of maps

$$(\alpha'_y, \alpha''_y) : F^*_{\mathfrak{p}}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \oplus I^*_{\mathfrak{p}}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \to \underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \oplus F^*_{\mathfrak{p}}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}},$$

where the morphisms α'_y, α''_y arise from the corresponding isogenies $F_{\mathfrak{p}}$ and $V_{\mathfrak{p}}$, respectively. In particular α'_y agrees with the map in Lemma 13.9.

By the crystal analogue of [9], Lem. 4.6, one therefore has the following commutative diagram on cohomology

$$\begin{tikzcd}
R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[r, equal] \arrow[d, "\mathrm{adj}"] & R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[d, "\mathrm{adj}"] \\
R^1 g_{\mathcal{K}!}\bar{\pi}_{2*}\bar{\pi}_2^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[r] \arrow[d, "\cong"', "(74)"] & R^1 g_{\mathcal{K}!}(F_{\mathfrak{p}*}F^*_{\mathfrak{p}} \oplus I_{\mathfrak{p}*}I^*_{\mathfrak{p}})\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[d, "\cong"'] \\
R^1 g_{\mathcal{K}'!}\bar{\pi}_2^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[r] \arrow[d, "R^1\alpha_y"] & R^1 g_{\mathcal{K}!}(F^*_{\mathfrak{p}} \oplus I^*_{\mathfrak{p}})\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[d, "R^1(\alpha'_y,\alpha''_y)"] \\
R^1 g_{\mathcal{K}'!}\pi_1^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[r] & R^1 g_{\mathcal{K}!}(\mathrm{id}^* \oplus F^*_{\mathfrak{p}})\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \\
R^1 g_{\mathcal{K}!}\pi_{1*}\pi_1^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[r] \arrow[u, "\cong", "(75)"'] \arrow[d, "\mathrm{trace}"] & R^1 g_{\mathcal{K}!}(\mathrm{id} \oplus F_{\mathfrak{p}*}F^*_{\mathfrak{p}})\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[u, "\cong"] \arrow[d, "\mathrm{trace}"] \\
R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \arrow[r, equal] & R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}}.
\end{tikzcd}$$

By Lemma 7.10, the trace map $F_{\mathfrak{p}*}F_{\mathfrak{p}}^* \to \mathrm{id}$ is zero. Therefore the operator $T_{\mathfrak{p}}$ is defined by the following sequence of morphisms:

$$R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \xrightarrow{\mathrm{adj}} R^1 g_{\mathcal{K}!}F_{\mathfrak{p}*}F_{\mathfrak{p}}^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \xrightarrow{\cong} R^1 g_{\mathcal{K}!}F_{\mathfrak{p}}^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \xrightarrow{R^1\alpha'_y} R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}}.$$

Finally note that the base change map $(\sigma^{d_{\mathfrak{p}}} \times \mathrm{id})^* R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \longrightarrow R^1 g_{\mathcal{K}!}F_{\mathfrak{p}}^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}}$ is defined as the adjoint of

$$R^1 g_{\mathcal{K}!}\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \longrightarrow (\sigma^{d_{\mathfrak{p}}} \times \mathrm{id})_* R^1 g_{\mathcal{K}!}F_{\mathfrak{p}}^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}} \xrightarrow{\cong} R^1 g_{\mathcal{K}!}F_{\mathfrak{p}*}F_{\mathfrak{p}}^*\underline{\mathcal{F}}^{(n)}_{\mathcal{K},\mathfrak{p}}$$

where the isomorphism on the right is the inverse to the isomorphism in the previous diagram in the middle. Because $\sigma^{d_{\mathfrak{p}}} \times \mathrm{id}$ is the identity, the assertion of the theorem is shown. ∎

14 Galois representations

The purpose of this section is to attach to any Drinfeld cuspidal eigenform f and any place v of K a continuous Galois representation $\rho_{f,v}\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_1(A_{f,v})$, for some unique $A_{f,v} \subset \mathbb{C}_v$ which is a finite extension of A_v and depends on f. This parallels the construction of Deligne in the case of classical modular forms. As in the classical situation, the connection between f and $\rho_{f,v}$ is given by the Eichler-Shimura relation, so that $\rho_{f,v}$ is uniquely determined by the Hecke eigenvalues of f.

It is unclear whether one should be able to have some kind of multiplicity one result for double cusp forms, by introducing a theory of new and old forms, cf. Example 15.4. For all cusp forms such a result is certainly wrong, because of the cuspidal eigenforms which are not doubly cuspidal, cf. Example 15.7. Examples for the latter, can also be obtained from [16]. In particular, it is not clear how to recover an eigenform f if $\rho_{f,v}$ is given.

We fix an admissible $\mathcal{K}$ and a place v and let $\mathfrak{p} = \mathfrak{p}_v$ denote the corresponding prime of A. By $\mathfrak{n}'$ we denote the product of $\mathfrak{p}$ with the minimal conductor $\mathfrak{n}$ of $\mathcal{K}$.

Let us first give some results on $\underline{\mathcal{S}}_n(\mathcal{K}) \otimes A/\mathfrak{p}^m$.

Proposition 14.1 *On* $\operatorname{Spec} A(\mathfrak{n}')$, *the* τ*-sheaf* $\underline{\mathcal{S}}_n(\mathcal{K}) \otimes A/\mathfrak{p}^m$ *is representable by a locally free* τ*-sheaf on which* τ *is an isomorphism. Furthermore, the étale sheaf* $(\underline{\mathcal{S}}_n(\mathcal{K}) \otimes A/\mathfrak{p}^m)_{\text{ét}}$ *is locally constant and locally free of rank* $d_{\mathfrak{p}} m s_n(\mathcal{K})$ *over* k.

Proof: In the proof of Proposition 13.4, we constructed a locally free τ-sheaf on $\operatorname{Spec} A(\mathfrak{n})$ which represents $\underline{\mathcal{S}}_n(\mathcal{K})$. Let $\underline{\mathcal{S}}'_{n,\mathfrak{p}^m}(\mathcal{K})$ be its restriction to $\operatorname{Spec} A(\mathfrak{n}')$ with coefficients changed to $A/\mathfrak{p}^m$. Because τ is a nil-isomorphism, we may replace $\underline{\mathcal{S}}'_{n,\mathfrak{p}^m}(\mathcal{K})$ as in loc. cit. by $\tau^j((\sigma^n \times \mathrm{id})^* \underline{\mathcal{S}}'_{n,\mathfrak{p}^m}(\mathcal{K}))$ for any $j > 0$, which is again locally free. If we choose j sufficiently large, then generically, τ will be injective. For later use, let us assume that the j here is larger than the j used to construct the torsion free representative $\underline{\mathcal{S}}_n(\mathcal{K})$ in Proposition 13.4. Thus by local freeness τ itself will be injective. Let us fix such a j and denote the corresponding τ-sheaf by $\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$.

By Remark 10.15, the dimension of $\mathfrak{b}_K^* \underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$ is $s_n(\mathcal{K})$, so that $\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$ is of rank $s_n(\mathcal{K})$. Let us show that τ is an isomorphism. This may be checked at stalks, i.e. for $\mathfrak{b}_{k_\mathfrak{q}}^* \underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$ for all $\mathfrak{q} \in \operatorname{Spec} A(\mathfrak{n}')$. By Lang's theorem this will follow, if we show that the τ-invariants of $\mathfrak{b}_{k_\mathfrak{q}^{\mathrm{alg}}}^* \underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$ have dimension $m s_n(\mathcal{K})[k_\mathfrak{p} : k]$ over k. But the module of τ-invariants is isomorphic to $(\mathfrak{b}_{k_\mathfrak{q}}^* \underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K}))_{\text{ét}}(k_\mathfrak{q}^{\mathrm{alg}})$ and its dimension can be computed using Theorem 10.12 due to Pink:

Consider $j_{\mathcal{K}!} \underline{\mathcal{F}}_{\mathcal{K}} \otimes_A A/\mathfrak{p}^m$ base changed to the fiber of $\overline{\mathfrak{M}}_{\mathcal{K}}$ above $\operatorname{Spec} k_\mathfrak{q}$. Because $\mathfrak{q}$ is prime to $\mathfrak{pn}$, the moduli schemes $\overline{\mathfrak{M}}_{\mathcal{K}\cap\mathcal{K}(\mathfrak{p}^m),k_\mathfrak{q}}$ are smooth over $\operatorname{Spec} k_\mathfrak{p}$. They are ordinary and the ramification of

$$\overline{\mathfrak{M}}_{\mathcal{K}\cap\mathcal{K}(\mathfrak{p}^m),k_\mathfrak{q}} \longrightarrow \overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}}$$

at the cusps is of p-power order. Also the étale sheaf $\underline{\mathcal{F}}_{\mathcal{K}} \otimes_A A/\mathfrak{p}^m$ describes the $\mathfrak{p}^m$-torsion points of the universal Drinfeld module over $\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{p}}$. In particular, it is of generic k-dimension $m d_\mathfrak{p}$. Hence by Theorem 10.12, the Euler-Poincaré characteristic of $\mathfrak{b}_{k_\mathfrak{q}}^* (j_{\mathcal{K}!} \underline{\mathcal{F}}_{\mathcal{K}} \otimes_A A/\mathfrak{p}^m)_{\text{ét}}$ is $s_n(\mathcal{K}) d_\mathfrak{p} m$. The only non-zero term when computing the Euler-Poincaré characteristic is the $H^1(\ldots)$-term. Because

$$H^1(\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{p}}, \mathfrak{b}_{k_\mathfrak{q}}^*, (j_{\mathcal{K}!} \underline{\mathcal{F}}_{\mathcal{K}} \otimes_A A/\mathfrak{p}^m)_{\text{ét}}) \cong (\mathfrak{b}_{k_\mathfrak{q}}^* \underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K}))_{\text{ét}},$$

we have thus shown that $(\mathfrak{b}_{k_\mathfrak{q}}^*\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K}))_{\text{ét}}$ is of the desired dimension. Furthermore our argument implies that $\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})_{\text{ét}}$ is locally free of dimension $ms_n(\mathcal{K})[k_\mathfrak{p} : k]$ over k, and that $\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$ is an étale φ-sheaf in the sense of [53], p. 770. Hence by [53], Prop. 6.1, the sheaf $\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})_{\text{ét}}$ is locally constant on $\operatorname{Spec} A(\mathfrak{n}')$. Thus we have proved all assertions. ∎

Because the j in the proof of the previous proposition is at least as big as that in Proposition 13.4, there is clearly a nil-isomorphism from $\underline{\mathcal{S}}_n(\mathcal{K}) \otimes_A A/\mathfrak{p}^m$ to $\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$. Therefore the saturation of the images of the $\underline{\mathcal{S}}_{n,i}(\mathcal{K}) \otimes_A A/\mathfrak{p}^m$ in $\mathfrak{b}_K^*\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$ define a filtration $\underline{\mathcal{S}}_{n,i,\mathfrak{p}^m}(\mathcal{K})$ on $\underline{\mathcal{S}}_{n,\mathfrak{p}^m}(\mathcal{K})$, such that all subquotients are locally free étale φ-sheaves. One may check that

$$\underline{\mathcal{S}}_{n,i}(\mathcal{K}) \otimes_A A/\mathfrak{p}^m \to \underline{\mathcal{S}}_{n,i,\mathfrak{p}^m}(\mathcal{K})$$

is a nil-isomorphism. Let us denote by $\overline{\underline{\mathcal{S}}}_{n,i,\mathfrak{p}^m}(\mathcal{K})$ the successive subquotients. They are Hecke-invariant, carry an action of A_i and are of rank $\operatorname{rank}_A A_i$ over $\operatorname{Spec} A(\mathfrak{n}') \otimes_A A/\mathfrak{p}^m$.

Definition 14.2 *By* $\underline{\mathcal{S}}_{n,i}^{\text{ét},v}(\mathcal{K})$*, we denote* $\varprojlim_m \underline{\mathcal{S}}_{n,i,\mathfrak{p}^m}(\mathcal{K})_{\text{ét}}$ *and by* $\overline{\underline{\mathcal{S}}}_{n,i}^{\text{ét},v}(\mathcal{K})$*, the inverse limit* $\varprojlim_m \overline{\underline{\mathcal{S}}}_{n,i,\mathfrak{p}^m}(\mathcal{K})_{\text{ét}}$.

As a corollary to the previous proposition, we obtain.

Corollary 14.3 *The étale sheaves* $\overline{\underline{\mathcal{S}}}_{n,i}^{\text{ét},v}(\mathcal{K})$ *are the subquotients of the filtration* $\underline{\mathcal{S}}_{n,i}^{\text{ét},v}(\mathcal{K})$ *of* $\underline{\mathcal{S}}_{n}^{\text{ét},v}(\mathcal{K})$*. They give rise to Galois representations*

$$\rho_{i,v} : \operatorname{Gal}(K^{\text{sep}}/K) \longrightarrow \operatorname{GL}_1(A_v \otimes_A A_i) \hookrightarrow \operatorname{GL}_{\operatorname{rank}_A A_i}(A_v),$$

which are unramified over $\operatorname{Spec} A(\mathfrak{n}')$.

Proof: Clearly the $\underline{\mathcal{S}}_{n,i,\mathfrak{p}^m}(\mathcal{K})$ give rise to a compatible system of Galois representations into $\operatorname{GL}_{\operatorname{rank}_A A_i}(A/\mathfrak{p}^m)$. Therefore it only remains to show that the resulting A_v-adic representation, which is denoted $\rho_{v,i}$, factors via $\operatorname{GL}_1(A_v \otimes_A A_i)$.

We know that the stalk at K^{alg} of $\overline{\underline{\mathcal{S}}}_{n,i}^{\text{ét},v}(\mathcal{K})$ is free over A_v of rank $\operatorname{rank}_A A_i$, and it carries an operation of A_i which commutes with the Galois action. Hence $\overline{\underline{\mathcal{S}}}_{n,i}^{\text{ét},v}(\mathcal{K})$ is an $(A_v \otimes_A A_i)[\operatorname{Gal}(K^{\text{alg}}/K)]$-module. Because $\overline{\underline{\mathcal{S}}}_{n,i}(\mathcal{K})$ is locally free over $K \otimes A_i$ of rank one, where the action of A_i arises from the Hecke action and that of K is the usual one, it easily follows that $\overline{\underline{\mathcal{S}}}_{n,i}^{\text{ét},v}(\mathcal{K})$ is locally free (and hence free) of rank one over $A_v \otimes_A A_i$. Thus the corollary is shown. ∎

Let us now consider the action of $\operatorname{Frob}_\mathfrak{q}$ under $\rho_{i,v}$ for some unramified place $\mathfrak{q} \in \operatorname{Spec} A(\mathfrak{n}')$, where $\operatorname{Frob}_\mathfrak{q}$ denotes a choice of geometric Frobenius in $\operatorname{Gal}(K^{\text{sep}}/K)$ at $\mathfrak{q}$.

Let $\operatorname{Frob}'_\mathfrak{q}$ be the arithmetic Frobenius $a \mapsto a^{q^{d_\mathfrak{q}}} \in \operatorname{Gal}(k_\mathfrak{q}^{\text{alg}}/k_\mathfrak{q})$. Because $\rho_{i,v}$ arises from the étale sheaf $\underline{\mathcal{S}}_{n}^{\text{ét},v}(\mathcal{K})$ over $\operatorname{Spec} A(\mathfrak{n}')$, the action of $\operatorname{Frob}_\mathfrak{q}$ under $\rho_{i,v}$ is the same as the action of $(\operatorname{Frob}'_\mathfrak{q})^{-1}$ on the stalk $\mathfrak{b}_\mathfrak{q}^*\underline{\mathcal{S}}_{n}^{\text{ét},v}(\mathcal{K})$ at $k_\mathfrak{q}^{\text{alg}}$. On $\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}^{\text{alg}}}$, one has

$$\left(\operatorname{id}_{\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}}} \times \operatorname{Frob}'_\mathfrak{q}\right)\left(\sigma_{\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}}} \times \operatorname{id}_{\operatorname{Spec} k_\mathfrak{q}^{\text{alg}}}\right) = \sigma_{\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}^{\text{alg}}}}.$$

Because $\sigma_{\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}^{\text{alg}}}}$ acts trivially on $H^1_{\text{ét}}(\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}^{\text{alg}}}, (j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}} \hat{\otimes} A_v)_{\text{ét},v}) \cong \underline{\mathcal{S}}_{n}^{\text{ét},v}(\mathcal{K})$, it follows that the action induced from $\sigma_{\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}}} \times \operatorname{id}_{\operatorname{Spec} k_\mathfrak{q}^{\text{alg}}} = F_\mathfrak{q} \times \operatorname{id}_{\operatorname{Spec} k_\mathfrak{q}^{\text{alg}}}$ on cohomology is the inverse of the action induced from $\operatorname{id}_{\overline{\mathfrak{M}}_{\mathcal{K},k_\mathfrak{q}}} \times \operatorname{Frob}'_\mathfrak{q}$. Therefore the former is equal to the Galois action of $\operatorname{Frob}_\mathfrak{q}$.

Theorem 14.4 (Eichler-Shimura relation) *The action of* $\mathrm{Frob}_{\mathfrak{q}}$ *and of* $T_{\mathfrak{q}}$ *on* $\underline{\mathcal{S}}_n^{\text{ét},v}(\mathcal{K})$ *agree. The same holds for the subquotients* $\overline{\underline{\mathcal{S}}}_{n,i}^{\text{ét},v}(\mathcal{K})$.

PROOF: By the above observations, it suffices to prove that on $\mathfrak{b}^*_{k_{\mathfrak{q}}^{\text{alg}}}\underline{\mathcal{S}}_n^{\text{ét},v}(\mathcal{K})$ the action of $F_{\mathfrak{q}} \times \mathrm{id}_{\mathrm{Spec}\, k_{\mathfrak{q}}^{\text{alg}}}$ agrees with that of $T_{\mathfrak{q}}$. An explicit description of the latter action is obtained from Theorem 13.10 by applying $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}$ and passing to inverse limits. The definition of $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}$, given in Subsection 7.3, implies that $(\alpha'_y)_{\text{ét}} = \mathrm{Frob}_{\mathfrak{q}}$, so that $T_{\mathfrak{q}}$ on $H^1_{\text{ét}}(\overline{\mathfrak{M}}_{\mathcal{K},k_{\mathfrak{q}}^{\text{alg}}}, (j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}\hat{\otimes}A_v)_{\text{ét},v})$ is the composition of the base change morphism

$$H^1(\overline{\mathfrak{M}}_{\mathcal{K},k_{\mathfrak{q}}^{\text{alg}}}, (j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}\hat{\otimes}A_v)_{\text{ét},v}) \longrightarrow H^1(\overline{\mathfrak{M}}_{\mathcal{K},k_{\mathfrak{q}}^{\text{alg}}}, F_{\mathfrak{q}}^*(j_{\mathcal{K}!}\underline{\mathcal{F}}_{\mathcal{K}}\hat{\otimes}A_v)_{\text{ét},v})$$

with the morphism $H^1(\mathrm{Frob}_{\mathfrak{q}} \times \mathrm{id})$. But this is precisely the definition of $\mathrm{Frob}_{\mathfrak{q}}$ on $H^1(\ldots)$, so that the theorem is proved. ∎

Corollary 14.5 *Let f be a cuspidal Drinfeld Hecke eigenform (over $\mathbb{C}_\infty$) and v a place of v. Let K_f denote the field generated over f by the Hecke eigenvalues $a_{\mathfrak{q}}$ of $T_{\mathfrak{q}}$ acting on f and let A_f be its ring of integers. We choose some isomorphism $\iota_v : \mathbb{C}_v \to \mathbb{C}_\infty$ which is the identity on K. Then there exists a place v' of A_f above v and a unique representation*

$$\rho_{f,v} : \mathrm{Gal}(K^{\text{sep}}/K) \longrightarrow \mathrm{GL}_1((A_f)_{v'})$$

for some choice $(A_f)_{v'} \subset \mathbb{C}_v$ such that $\rho_{f,v}$ is unramified above $\mathrm{Spec}\, A(\mathfrak{n}')$ *and such that for all $\mathfrak{q}$ prime to $\mathfrak{n}'$, one has*

$$\iota_v(\rho_{f,v}(\mathrm{Frob}'_{\mathfrak{q}})) = a_{\mathfrak{p}}.$$

PROOF: The existence follows from the previous theorem and the correspondence between conjugacy classes of forms f_j on the one hand and crystals $\underline{\mathcal{S}}_{n,i}(\mathcal{K})$ on the other: When defining the f_j, it was observed that any eigenform f has the same Hecke eigenvalues as one of the f_j. Thus to each f one can attach a crystal $\overline{\underline{\mathcal{S}}}_{n,i}(\mathcal{K})$ with the same Hecke action. This crystal gives rise to the representation $\rho_{v,i}$ of the previous theorem, in which the eigenvalues of Frobenius are determined by the Hecke operation on $\underline{\mathcal{S}}_{n,i}^{\text{ét},v}(\mathcal{K})$.

Say $a'_{\mathfrak{p}}$ is the image of $T_{\mathfrak{p}}$ under $A_i \mapsto A_v \otimes_A A_i$. These eigenvalues $a'_{\mathfrak{p}}$ are algebraic over A, and so are the eigenvalues for the Hecke operators $S_{\mathfrak{m}}$. Together they generate $A_v \otimes_A A_i$ over A_v. Note that A_f is the normal closure of A_i and hence $A_i \otimes_A A_v \subset A_i \otimes_A A_v$. Hence there is a unique embedding $(A_f)_{v'} \hookrightarrow \mathbb{C}_v$ such that after composition with ι_v the image of $a'_{\mathfrak{p}}$ equals $a_{\mathfrak{p}}$. The component $\mathrm{Gal}(K^{\text{sep}}/K) \to \mathrm{GL}_1((A_f)_{v'})$ corresponding to v' of $\rho_{v,i}$ has all the required properties. Finally, having shown existence the uniqueness is clear by the Cebotarov density theorem. ∎

Remark 14.6 In the classical situation, the Galois representations attached to modular forms are two-dimensional. Basically this comes from complex conjugation acting on the Hodge decomposition of $H^1(\ldots)$. As there is no such decomposition on the corresponding $H^1(\ldots)$ in the function field case (the $H^{1,0}$-term is missing), one-dimensionality should not come as a surprise.

Since the Hecke action leaves the kernel of $\underline{\mathcal{S}}_n(\mathcal{K}) \to \underline{\mathcal{S}}_n^2(\mathcal{K})$ invariant, it is possible to completely determine all Galois representations corresponding to this kernel. After coefficient and base change to K this kernel is given as

$$\coprod_{x \in K^* B(\mathbb{A}^f)\backslash \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}} \mathbb{1}_{\mathfrak{N}_K^\infty/(B(\mathbb{A}^f)\cap x^{-1}\mathcal{K}x),K} \otimes_A P_x^{\otimes(n-2)},$$

for suitable A-modules P_x of rank one, if $n > 2$, cf. Lemma 2.11 and Theorem 11.19. In the case $n = 2$ it is the quotient of this module by some number of unit τ-sheaves, cf. Corollary 12.8. The scheme $\mathfrak{N}_K^\infty/(B(\mathbb{A}^f) \cap x^{-1}\mathcal{K}x)$ parametrizes rank one Drinfeld modules over K with a level structure depending on x, i.e., by results of Drinfeld it is the spectrum of a finite abelian extension field K_x of K which is totally split at ∞. The Galois action is then simply the action on K_x. It follows that from the cusp x, one obtains the left regular representation

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(K_x/K) \twoheadrightarrow K[\mathrm{Gal}(K_x/K)].$$

After replacing the coefficient field K of $K[\mathrm{Gal}(K_x/K)]$ by suitable extensions, the representation clearly has a decomposition series into one-dimensional characters of finite order. We have shown:

Corollary 14.7 *Let the notation be as above. Then for $n > 2$, the Galois representations of Corollary 14.5 which arise from the kernel of $\underline{\mathcal{S}}_n(\mathcal{K}) \to \underline{\mathcal{S}}_n^2(\mathcal{K})$ in the four term sequence (72), are precisely the one-dimensional characters that appear in a composition series of*

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(K_x/K) \longrightarrow K^{\mathrm{alg}}[\mathrm{Gal}(K_x/K)]$$

for some $x \in K^* B(\mathbb{A}^f)\backslash \mathrm{GL}_2(\mathbb{A}^f)/\mathcal{K}$.

For $n = 2$, one needs to remove from the above list one copy of the trivial representation for each connected component of $\mathfrak{M}_{\mathcal{K},\mathbb{C}_\infty}$, i.e., $d(\mathcal{K})$-many such copies.

As pointed out in Remark 12.9, we hope that the corresponding eigenforms can be given explicitly by using Poincaré series. Also, if desired, one can make the extensions K_x of K totally explicit, so that one has a precise description of the Galois representations which occur.

On the other hand, the examples in the following section suggest that the one-dimensional representations that arise from double cusp forms are typically not of finite order. However there is clearly one exception, as can be seen from the following result, which is also implicitly contained in [17].

Theorem 14.8 *There is a canonical isomorphism*

$$\underline{\mathcal{S}}_2^2(\mathcal{K}) \cong (P_\tau^{\text{ét}}\, R^1_{\text{ét}}\bar{g}_{\mathcal{K}*}\mathbb{1}^{\text{ét}}_{\overline{\mathfrak{M}}_{\mathcal{K}},k}) \otimes_k A,$$

where $P_\tau^{\text{ét}}$ is as in Theorem 7.17.

PROOF: Clearly the maximal extension of $\underline{\mathcal{F}}_{\mathcal{K}}$ to $\overline{\mathfrak{M}}_{\mathcal{K}}$ is $\mathbb{1}_{\overline{\mathfrak{M}}_{\mathcal{K}},A}$, which in turn is obtained from $\mathbb{1}_{\overline{\mathfrak{M}}_{\mathcal{K}},k}$ by changing coefficients from k to A. Therefore we have

$$\underline{\mathcal{S}}_2^2(\mathcal{K}) \cong R^1\bar{g}_{\mathcal{K}*}\mathbb{1}_{\overline{\mathfrak{M}}_{\mathcal{K}},k} \otimes_k A \overset{\text{Thms. 7.17 and 7.18}}{\cong} (P_\tau^{\text{ét}}\, R^1_{\text{ét}}\bar{g}_{\mathcal{K}*}\mathbb{1}^{\text{ét}}_{\overline{\mathfrak{M}}_{\mathcal{K}},k}) \otimes_k A,$$

because any étale sheaf on $\operatorname{Spec} K$ is locally constant. The assertion follows. ∎

15 An example

Throughout, we fix $A = k[T]$ and let $\mathcal{K} = \mathcal{K}(\mathfrak{n})$ for the level $\mathfrak{n} = (T)$. So we are in the same set-up as in Example 6.13. In this case $\mathrm{Cl}_{\mathcal{K}}$ is trivial. In the sequel we want to compute the moduli space of Drinfeld-modules with a level T-structure. A good general reference for such computations is [46].

The calculations below are joint work with R. Pink.

Say we want 1 and s as a basis of T-torsion points of the Drinfeld-module given by $\varphi_T(x) = a_0 x + a_1 x^q + a_2 x^{q^2}$. The fact that all k-linear combinations of $1, s$ will be roots of the above polynomial means that we have

$$\begin{vmatrix} x & x^q & x^{q^2} \\ s & s^q & s^{q^2} \\ 1 & 1 & 1 \end{vmatrix} c = \varphi_T(x)$$

for some unit c. Because $a_0 = \theta$, we must have $c = \theta / \det \left(\begin{smallmatrix} s^q & s^{q^2} \\ 1 & 1 \end{smallmatrix} \right)$. This yields

$$\varphi_T(x) = \theta\Big(x + \frac{s^{q^2} - s}{s^q - s^{q^2}} x^q + \frac{s - s^q}{s^q - s^{q^2}} x^{q^2}\Big).$$

It follows that $\mathfrak{M}_{\mathcal{K}}$ is the spectrum of the ring $R' = k[\theta^{\pm 1}, s, (s - s^q)^{-1}]$.

The associated τ-module is given as $R'\{\tau\}$ where the generator T of A acts via φ_T. Therefore the elements $1, \tau$ form a basis of $R'\{\tau\}$ over $R'[T]$. If we set $u = s - s^q$, then the morphism τ is given by

$$\tau = \begin{pmatrix} 0 & \frac{T - a_0}{a_2} \\ 1 & \frac{-a_1}{a_2} \end{pmatrix} (\sigma \times \mathrm{id}) = \begin{pmatrix} 0 & \Big(\frac{T}{\theta} - 1\Big) u^{1-q} \\ 1 & 1 + u^{1-q} \end{pmatrix} (\sigma \times \mathrm{id}).$$

We define $R := k[\theta^{\pm 1}, u^{\pm 1}]$. This parametrizes the moduli scheme for rank two Drinfeld-modules with a level $\mathcal{K}'$-structure, where $\mathcal{K}' \supset \mathcal{K}$ and the T-component of $\mathcal{K}'$ is given by $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(A_{(T)}) : a, d, 1 + c \equiv 1 \pmod{T}\}$. In particular one of the T-torsion points, namely 1, is fixed for the moduli problem. The coordinate u gives a monic equation of degree q for the second torsion point s.

From now on, we only work over $Y = \operatorname{Spec} R$ and our τ-sheaf $\underline{\mathcal{F}}$ is given by $(\mathcal{O}^2_{X \times \operatorname{Spec} A}, \tau)$, where τ is as above. Because we have a level T-structure, we consider Y as a scheme over $X := \operatorname{Spec} A((\theta)) = \operatorname{Spec} k[\theta, \theta^{-1}]$. In terms of u, the universal Drinfeld-module is given by $\varphi_T = \theta\Big(1 + (-1 - u^{1-q})\tau + u^{1-q}\tau^2\Big)$. Clearly for $u \to \infty$, we have a cusp and the module degenerates to a module of rank 1. After applying the isogeny $u \in R^*$, it is easy to see, that $u \to 0$ is also a cusp. Drinfeld's compactification is now $\bar{Y} := \mathbb{P}^1_{\operatorname{Spec} k[\theta^{\pm 1}]}$. Let $\bar{g} \colon \bar{Y} \to X$ be the structure morphism.

Set $\overline{\mathcal{F}} := \mathcal{O}_{\bar{Y} \times \operatorname{Spec} A}(-\infty)^2$. One verifies that τ extends to this sheaf and furthermore that τ has the following specializations:

$$\begin{aligned} \tau &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} (\sigma \times \mathrm{id}) \quad \text{at } u = 0 \\ \tau &= u \begin{pmatrix} 0 & (T/\theta - 1) u^{q-1} \\ 1 & u^{q-1} \end{pmatrix} u^{-q} \Big|_{u=\infty} (\sigma \times \mathrm{id}) = \begin{pmatrix} 0 & T/\theta - 1 \\ 0 & 1 \end{pmatrix} (\sigma \times \mathrm{id}) \quad \text{at } u = \infty. \end{aligned}$$

In both cases, the non-nilpotent part has rank 1. Therefore the τ-sheaf $\overline{\underline{\mathcal{F}}} := (\mathcal{O}_{\bar{Y} \times \operatorname{Spec} A}(-\infty), \tau)$ is the maximal extension of $\underline{\mathcal{F}}$.

Proposition 15.1 *The crystal of Drinfeld modular double cusp forms for $\mathcal{K}'$ is given by $\underline{\mathcal{S}}^2_{n+2}(\mathcal{K}') := R^1 \bar{g}_* \operatorname{Sym}^n \overline{\underline{\mathcal{F}}}$.*

Because the underlying sheaf of $\overline{\mathcal{F}}$ is of pullback type, it follows that the τ-sheaf $\underline{\mathcal{S}}^2_{n+2}(\mathcal{K}')$ is free over $\operatorname{Spec} k[\theta^{\pm 1}]$ of rank $\dim_k H^1(\mathbb{P}^1, \mathcal{O}(-n\infty)^{n+1}) = n^2 - 1$. On the other hand, the expected rank of a good representative of the crystal $\underline{\mathcal{S}}^2_n(\mathcal{K}')$, is given by

$$s^2_n(\mathcal{K}') = \begin{cases} g(\bar{Y}_K) = 0 & \text{if } n = 2 \\ (n-2)(g(\bar{Y}_K) + h(\bar{Y}_K)) + g(\bar{Y}_K) - 1 = (n-2) - 1 & \text{if } n > 2 \end{cases}$$

In particular, for $n < 4$, the crystal $\underline{\mathcal{S}}^2_n(\mathcal{K}')$ is zero.

Define $\alpha(n)$ as the n-th symmetric power of the matrix $\alpha := \begin{pmatrix} 0 & b \\ 1 & c \end{pmatrix} := \begin{pmatrix} 0 & (T/\theta - 1)u^{q-1} \\ 1 & u^{q-1} \end{pmatrix}$. Then $\alpha(n)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & b^n \\ 0 & 0 & 0 & \dots & b^{n-1} & \binom{n}{n-1} b^{n-1} c \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & b^2 & \dots & \binom{n-1}{2} b^2 c^{n-3} & \binom{n}{2} b^2 c^{n-2} \\ 0 & b & 2bc & \dots & \binom{n-1}{1} bc^{n-2} & \binom{n}{1} b^1 c^{n-1} \\ 1 & c & c^2 & \dots & \binom{n-1}{0} c^{n-1} & \binom{n}{0} c^n \end{pmatrix}$$

For the actual computation of $\underline{\mathcal{S}}^2_{n+2}(\mathcal{K}')$, let $m = (q-1)(n-1) - 1$ and consider the following diagram for $n \geq 2$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\bar{Y}\times A}(-n\infty)^{n+1} \arrow[r] \arrow[d, "\sigma"] & \mathcal{O}_{\bar{Y}\times A}(-\infty)^{n+1} \arrow[r] \arrow[d, "\sigma"] & \frac{\mathcal{O}_{\bar{Y}\times A}(-\infty)^{n+1}}{\mathcal{O}_{\bar{Y}\times A}(-n\infty)^{n+1}} \arrow[r] \arrow[dd, "\tau"] & 0 \\ & \mathcal{O}_{\bar{Y}\times A}(-qn\infty)^{n+1} \arrow[r] \arrow[d, "\alpha(n)"] & \mathcal{O}_{\bar{Y}\times A}(-q\infty)^{n+1} \arrow[d, "\alpha(n)"] & & \\ 0 \arrow[r] & \mathcal{O}_{\bar{Y}\times A}(-n\infty)^{n+1} \arrow[r] & \mathcal{O}_{\bar{Y}\times A}(m\infty)^{n+1} \arrow[r] & \frac{\mathcal{O}_{\bar{Y}\times A}(m\infty)^{n+1}}{\mathcal{O}_{\bar{Y}\times A}(-n\infty)^{n+1}} \arrow[r] & 0. \end{tikzcd}$$

Applying the long exact sequence of cohomology yields a commutative diagram

$$\begin{tikzcd} \frac{\mathcal{O}_{\bar{Y}\times A}(-\infty)^{n+1}}{\mathcal{O}_{\bar{Y}\times A}(-n\infty)^{n+1}} \arrow[r, "\cong"] \arrow[d, "\tau"] & R^1\bar{g}_*\mathcal{O}_{\bar{Y}\times A}(-n\infty)^{n+1} \arrow[d, "R^1\bar{g}_*(\tau)"] & \\ \frac{\mathcal{O}_{\bar{Y}\times A}(m\infty)^{n+1}}{\mathcal{O}_{\bar{Y}\times A}(-n\infty)^{n+1}} \arrow[r] & R^1\bar{g}_*\mathcal{O}_{\bar{Y}\times A}(-n\infty)^{n+1} \arrow[r] & 0 \end{tikzcd}$$

As a basis of $\frac{\mathcal{O}_{\bar{Y}\times A}(-\infty)}{\mathcal{O}_{\bar{Y}\times A}(-n\infty)}$ over $k[\theta^{\pm 1}]$ we take $u^{-(n-1)}, \dots, u^{-1}$. The standard basis of the free module $\mathcal{O}^{n+1}_{\bar{Y}\times A}$ we denote by $v_1, \dots, v_{n+1}$. Then a basis of the top left term is given by

$$\{u^{-i} v_j : i \in \{1, \dots, n-1\},\, j \in \{1, \dots, n+1\}\}.$$

Let us now compute $\underline{\mathcal{S}}^2_n(\mathcal{K}')$ explicitly in the simplest non-trivial case, i.e., for $n = 4$. Then our basis is $\{u^{-1} v_i : i = 1, 2, 3\}$. The image $\tau(u^{-1} v_i)$ is the residue with respect to u of the i-th column in $\alpha(2)(\sigma \times \mathrm{id})u^{-1}$. Hence the images are the columns in

$$\begin{aligned}
&\mathrm{Res}_u \begin{pmatrix} 0 & 0 & b^2 \\ 0 & b & 2bc \\ 1 & c & c^2 \end{pmatrix} u^{-q} \\
&= \mathrm{Res}_u \begin{pmatrix} 0 & 0 & (T/\theta-1)^2 u^{q-2} \\ 0 & (T/\theta-1)u^{-1} & 2(T/\theta-1)u^{-1}(1+u^{q-1}) \\ 1 & u^{-q}+u^{-1} & u^{-q}(1+u^{q-1})^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & (T/\theta-1) & 2(T/\theta-1) \\ 0 & 1 & 2 \end{pmatrix}.
\end{aligned}$$

On the subcrystal generated by v_1 and $2v_2 - v_3$, the map τ is zero. A basis of the quotient is given by the image $\bar{v}_2$ of v_2, which is mapped to

$$(0, T/\theta - 1, 1)^t \pmod{\langle v_1, 2v_2 - v_3\rangle} = (T/\theta + 1)\bar{v}_2,$$

where v^t is the transpose of a vector v. Thus we have shown:

Proposition 15.2 $\underline{\mathcal{S}}_4^2(\mathcal{K}') \cong (\mathcal{O}_{X\times \operatorname{Spec} A}, (T/\theta+1)(\sigma \times \mathrm{id}))$.

Using similar, but more sophisticated calculations, one can show the following:

Proposition 15.3 *a) If $q \geq n \geq 2$, then*

$$\underline{\mathcal{S}}_{n+2}^2(\mathcal{K}') \cong \bigoplus_{j=1}^{n-1} \Big(\mathcal{O}_{X\times \operatorname{Spec} A}, \sum_{\mu=0}^{n} \binom{\mu}{j}\binom{j}{n-\mu}(T/\theta-1)^{n-\mu}(\sigma\times \mathrm{id})\Big).$$

b) For $q = 2$, one has

$$\underline{\mathcal{S}}_5^2(\mathcal{K}') \cong \Big(\mathcal{O}^2_{X\times \operatorname{Spec} A}, \left(\begin{smallmatrix} 1 & T/\theta \\ T^2/\theta & 1 \end{smallmatrix}\right)(\sigma\times \mathrm{id})\Big).$$

c) For $q = 3$, one has

$$\underline{\mathcal{S}}_6^2(\mathcal{K}') \cong \Big(\mathcal{O}^2_{X\times \operatorname{Spec} A}, \left(\begin{smallmatrix} 1 & T/\theta \\ T^3/\theta & 1 \end{smallmatrix}\right)(\sigma\times \mathrm{id})\Big) \oplus (\mathcal{O}_{X\times \operatorname{Spec} A}, (T/\theta-1)^2(\sigma\times \mathrm{id})).$$

Example 15.4 The following shows that no naïve multiplicity one theorem can hold for doubly cuspidal Drinfeld modular forms. For $q > 2$ part a) above implies that $\underline{\mathcal{S}}_5^2(\mathcal{K}') \cong (\mathcal{O}_{X\times \operatorname{Spec} A}, (2T/\theta+1)(\sigma\times\mathrm{id}))^2$. Since the Hecke action on cusp forms is determined by the Galois representation which in turn is completely determined by the simple pieces of $\underline{\mathcal{S}}_n^2(\mathcal{K}')$, there exist two linearly independent doubly cuspidal Hecke eigenforms for $\mathcal{K}'$ which have the same systems of Hecke eigenvalues.

Consider the direct sum decomposition of $\underline{\mathcal{S}}_{n+2}^2(\mathcal{K}')$ in Proposition 15.3 a). The degree in T of the polynomial $\sum_{\mu=0}^{n} \binom{\mu}{j}\binom{j}{n-\mu}(T/\theta-1)^{n-\mu}$ is given by $\min\{j, n-j\}$. This degree can be thought of as the weight of the crystal at $T=\infty$. On the other hand, the weight of $\mathrm{Sym}^n\, \underline{\overline{\mathcal{F}}}$ at ∞ is given by $n/2$. Because for all but at most one j one has $\min\{j, n-2-j\} < n/2$, we have shown:

Corollary 15.5 *Weights are not preserved under $R^1\bar{g}_*$.*

Let us now compute the Hecke eigenvalues of the crystal $\underline{\mathcal{S}}_4^2(\mathcal{K}')$, using the Eichler-Shimura relation, Theorem 13.10. Let $\mathbf{p}(\theta)$ be an irreducible polynomial in $k[\theta]$ which is prime to θ, say of degree d, and which satisfies $\mathbf{p}(0) = 1$. If θ' in k^{alg} is a root of $\mathbf{p}$, then we have

$$\mathbf{p}(\theta) = (1 - \frac{\theta}{\theta'})(1 - \frac{\theta}{\theta'^q}) \ldots (1 - \frac{\theta}{\theta'^{q^{d-1}}}).$$

Let f be a non-zero double cusp form of weight 4 for $\mathcal{K}'$. The space of all such is one-dimensional, because $\underline{\mathcal{S}}_4^2(\mathcal{K}')$ is representable by a rank 1 τ-sheaf with injective τ, and thus f must be a Hecke eigenform.

To compute its Hecke eigenvalue $a_{(\mathbf{p})}$, following the recipe given in Theorem 13.10 one needs to compute the action of τ^d on the reduction of $\underline{\mathcal{S}}_4^2(\mathcal{K}')$ at $(\mathbf{p})$. We have

$$\tau^d = (1 + T/\theta)(1 + T/\theta^q) \ldots (1 + T/\theta^{q^{d-1}})(\sigma^d \times \mathrm{id}).$$

In $k[\theta]/(\mathbf{p}(\theta)) \otimes_k k[T]$, this becomes $\mathbf{p}(-T) \in k[\theta]/(\mathbf{p}(\theta)) \otimes_k k[T]$. We have shown:

Proposition 15.6 *Let $\mathbf{p} \in k[T]$ be irreducible and normalized such that $\mathbf{p}(0) = 1$. Then for all $f \in \mathbf{S}(\mathcal{K}')$ one has*

$$T_{(\mathbf{p})} f = \mathbf{p}(-T) f,$$

where $\mathbf{p}(-T) \in \mathbb{C}_\infty$ via $\iota_{\mathbb{C}_\infty} : k[T] \to \mathbb{C}_\infty$.

Example 15.7 Let us conclude by computing the Hecke action on the quotients of the cusp forms by the double cusp forms. The crystal to consider on $\bar{Y}$ is the unit crystal $\underline{\mathbb{1}}_{(\{0\}\times \operatorname{Spec} A)\times \operatorname{Spec} A, A} \oplus \underline{\mathbb{1}}_{(\{\infty\}\times \operatorname{Spec} A)\times \operatorname{Spec} A, A}$, supported on the cusps. If we compute the pushforward under $\bar{Y} \to X$, this simply yields the crystal $\underline{\mathbb{1}}^2_{\operatorname{Spec} A, A}$. A trivial calculation shows that all Hecke eigenvalues are identically one for the Hecke operators $T_{(\mathbf{p})}$.

Let us compare this result with the computation of the Hecke eigenvalues of the Poincaré series $P_{q+1,1} \in \mathbf{S}_{q+1,1}(\mathrm{GL}_2(\hat{A}))$ in [16]. There it was shown that $T'_{\mathfrak{p}} P_{q+1,1} = p'_{\mathfrak{p}} P_{q+1,1}$ for all primes $\mathfrak{p}$, where $T'_{\mathfrak{p}}$ and $p'_{\mathfrak{p}}$ are as in Example 6.13. Because $P_{q+1,1}$ is not doubly cuspidal, by the above computation its associated crystal must be the unit crystal, and hence via the Eichler-Shimura correspondence we must have $T_{\mathfrak{p}} P_{q+1,1} = P_{q+1,1}$. This is consistent with the computations in Example 6.13, which assert that $T'_{\mathfrak{p}} = p'_{\mathfrak{p}} T_{\mathfrak{p}}$.

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