Arithmetic over Function Fields (a Cohomological Approach)

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1 Introduction

The present article intends to be a survey of some recent developments on a particular aspect in the arithmetic of function fields, in which the present author was and still is involved. It is not intended to be a survey on all recent developments, of which there are many, nor on all the foundations of the subject, for which there is a number of good references available, such as [Al96], [Ge86], [Go96]. The main emphasis, as expressed by the subtitle is to advertise some developments that are based on a cohomological theory, introduced by R. Pink and the author, [BP04].

This survey is aimed at a reader who has some familiarity with the arithmetic of elliptic curves over number fields, with algebraic geometry and étale sheaves, and has perhaps some (vague) ideas about motives, *and* who wants to learn more about parallel aspects in the arithmetic of functions fields.

Our starting point in Section 2 is a short review of the similarities between elliptic curves on the one and Drinfeld modules on the other hand. We emphasize motivically interesting information that is encoded in them, namely their analytic and étale realizations. The subsequent section, gives a rapid introduction into some aspects of Anderson's theory of t-motives. It generalizes the theory of Drinfeld modules and thereby provides some additional flexibility.

Section 4 introduces the cohomological viewpoint introduced by R. Pink and the author, cf. [BP04]. It is a natural generalization of Anderson's theory. The construction starts with a theory that looks very much like the theory of coherent sheaves, where as an additional piece of data the sheaves are equipped with an endomorphism. (The endomorphism itself needs the absolute Frobenius endomorphism which is present in the characteristic p situation we consider. A similar type of object in characteristic zero would be a vector bundle with a connection.)

Then comes a crucial point. We localize this first category at a suitable multiplicatively closed subset. The resulting category is called the 'category of crystals over function fields'. The reader should not be deterred by this two step construction, but instead realize all the improvements of the theory that result from the localization. The motivation for the construction is that in [BP04] we realized that in the localized category there would exist a functor 'extension by zero' which is not present in the original category. The introduction of crystals is perhaps the main novelty in comparison to some earlier concepts generalizing Drinfeld modules.

As a test case, we compare the theory of crystals over $\mathbb{F}_p$ with the theory of constructible étale sheaves of $\mathbb{F}_p$-modules over schemes of characteristic p, and establish an equivalence of categories.

The following section gives two applications of the cohomological theory. The first is a rationality proof for L-functions that can be attached to families of t-motives or of Drinfeld modules. An analytic proof had previously been given by Taguchi and Wan, cf. [TW96], using Dwork style methods. A similar development had taken place in the rationality proofs of L-functions of varieties over finite fields. There again, it was first Dwork who gave an analytic and then Grothendieck who gave an algebraic proof. It was precisely this parallel that motivated the construction of the theory of crystals and led to [BP04].

The second application in Section 5 is the proof of the existence of a meromorphic continuation of global L-functions attached to t-motives by Goss to a mod p analog of the complex plane. Here again, the poineering work [TW96] of Taguchi and Wan had yielded a first p-adic analytic proof (at least in an important special case).

In the last section, we explain yet another application, namely the construction of something that could be called a motive for Drinfeld modular forms. This 'motive' is an arithmetic object whose analytic realization is the space of Drinfeld cusp forms for a fixed weight and level. Its étale realization allows one to attach Galois representations to Drinfeld cusp forms.

There are many interesting open problems in this subject, and throughout the last two sections, we describe a number of these. It is hoped that they will further stimulate the interest in the arithmetic of function fields.

Much of the work presented here, even that which did not involve a direct collaboration with R. Pink, nevertheless was influenced by his comments, ideas and interest, and it is a great pleasure to thank him for this. There is a large number of people whose work was foundational, inspirational, or directly related to the contents of this survey, and that I would like to mention here in the introduction, namely, in alphabetical order, G. W. Anderson, V.-G. Drinfeld, F. Gardeyn, E.-U. Gekeler, D. Goss. U. Hartl, Y. Taguchi, D. Wan.

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2 The basic example

Let us first recall some arithmetic properties of an elliptic curve E over $\mathbb{Q}$ which are basic for its realization as a motive.

1. Via $\mathbb{Q} \hookrightarrow \mathbb{C}$ the curve E becomes an elliptic curve over $\mathbb{C}$. Its first Betti homology is $\Lambda := H_1(E(\mathbb{C}), \mathbb{Z})$, and one has a pairing
$$\Lambda \times H^0(E(\mathbb{C}), \Omega_{E(\mathbb{C})/\mathbb{C}}) \to \mathbb{C} : (\lambda, \omega) \mapsto \int_\lambda \omega$$
so that $E(\mathbb{C}) \cong \mathrm{Hom}_\mathbb{C}(H^0(E(\mathbb{C}), \Omega_{E(\mathbb{C})/\mathbb{C}}), \mathbb{C})/\Lambda \cong \mathbb{C}/\Lambda$.
2. Furthermore for any rational prime ℓ of $\mathbb{Z}$ one has the ℓ-adic Tate module $\mathrm{Tate}_\ell(E)$ of E. As a group it is isomorphic to $\mathbb{Z}_\ell^2$, and it provides us with a representation of the absolute Galois group $G_\mathbb{Q} := \mathrm{Gal}(\mathbb{Q}^{\mathrm{sep}}/\mathbb{Q})$ of $\mathbb{Q}$ on $\mathrm{Tate}_\ell(E)$, i.e., a homomorphism $\rho_{E,\ell} \colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$.

The Betti cohomology of E may be viewed as its *analytic realization*, the Tate module of E as its *ℓ-adic étale realization*. (The motive of E in the sense of Grothendieck also contains parts in degree 0 and 2 – but these are not particular to E, and we therefore do not consider these.)

Let $\mathbb{F}_q$ be the finite field of q elements, and let F be a function field of transcendence degree one over its constant field $\mathbb{F}_q$. We fix a place, denoted by ∞, of F, and let A denote the ring of functions in F which have a pole at most at ∞. Then A is a Dedekind domain.

The completion F_∞ of F at ∞ is a discretely valued field. By q_∞ the cardinality of its residue field is denoted and by val_∞ its normalized valuation. The latter extends uniquely to a valuation on the algebraic closure F_∞^{alg} of F_∞. The completion $\mathbb{C}_\infty$ of F_∞^{alg} with respect to val_∞ remains algebraically closed. Denoting the extended valuation again by val_∞, the expression $|.|_\infty := q_\infty^{-\mathrm{val}_\infty}$ defines a norm on $\mathbb{C}_\infty$ which is often abbreviated $|.|$. Finally, ι denotes the composite homomorphism $A \hookrightarrow F \hookrightarrow F_\infty \hookrightarrow \mathbb{C}_\infty$.

We fix a finite extension L of F and a Drinfeld A-module ϕ of rank r on L of characteristic $\iota_L \colon A \hookrightarrow F \hookrightarrow L$. (Below we recall these and some further definitions.) To ϕ one associates:

1. Its *analytic realization*, which is a discrete A-lattice $\Lambda \subset \mathbb{C}_\infty$ of rank r, such that ϕ base changed to $\mathbb{C}_\infty$ arises from Λ.
2. For every place v of L which is not above ∞, its *v-adic étale realization*, which is a representation of the absolute Galois group G_L of L on the v-adic Tate module $\mathrm{Tate}_v(\phi)$ of ϕ. If A_v is the completion of A at v, then $\mathrm{Tate}_v(\phi)$ is isomorphic to A_v^r. The action of G_L yields an A_v-linear representation on $\mathrm{Tate}_v(\phi)$.

We owe, at least to the novice in function field arithmetic, some definitions. For any ring R over $\mathbb{F}_q$ one defines $R\{\tau\}$ as the non-commutative polynomial ring subject to the non-commutation rule $\tau r = r^q\tau$ for any $r \in R$. An alternative description of $R\{\tau\}$ is as follows. Let $R[z]_{\mathbb{F}_q}$ denote the set of $\mathbb{F}_q$-linear polynomials in the (commutative) polynomial ring $R[z]$, i.e., of polynomials of the form $\sum_i \beta_i z^{q^i}$. These are precisely the polynomials p with $p(\alpha x+y) = \alpha p(x)+p(y)$ for all $\alpha \in \mathbb{F}_q$ and all x, y in some R-algebra, i.e., they define $\mathbb{F}_q$-linear maps on R-algebras. Then $R[z]_{\mathbb{F}_q}$ is a ring under addition and composition of polynomials. Moreover the substitution $\tau^i \mapsto z^{q^i}$ yields an isomorphism of rings $R\{\tau\} \longrightarrow R[z]_{\mathbb{F}_q}$. In the sequel, we will always use a small Greek letter to denote a polynomial in $R\{\tau\}$, and the corresponding capital one to denote its image in $R[z]_{\mathbb{F}_q}$ – for instance $\phi_a \leftrightarrow \Phi_a$.

A *Drinfeld A-module ϕ on some field K* is a ring homomorphism $\phi\colon A \to K\{\tau\} : a \mapsto \phi_a$ such that its image $\phi(A)$ contains some non-constant polynomial of $K\{\tau\}$. There is a unique positive integer $r \in \mathbb{N}$, called the *rank of ϕ* such that for any $a \in A$ the highest non-zero coefficient of ϕ_a occurs in degree $r \deg(a)$.

The composition $\iota_K : A \to K$ of ϕ with the projection $K\{\tau\} \to K$ onto the zeroth coefficient is a ring homomorphism which is called the *characteristic of ϕ*. The Drinfeld A-module ϕ is called of *generic characteristic* if ι_K is injective. Otherwise it is called of *special characteristic*. The case of generic characteristic is the one analogous to that of an elliptic curve over a number field. The other case corresponds to the case of an elliptic curve over a field of positive characteristic.

If $K \hookrightarrow \mathbb{C}_\infty$ is a field homomorphism, the base change of ϕ to $\mathbb{C}_\infty$ is obtained by composing ϕ with the induced monomorphism $K\{\tau\} \hookrightarrow \mathbb{C}_\infty\{\tau\}$. We will only consider homomorphisms $\alpha : K \hookrightarrow \mathbb{C}_\infty$ such that $\alpha\iota_K$ agrees with the homomorphism $\iota\colon A \to \mathbb{C}_\infty$ defined above.

A *discrete A-lattice* $\Lambda \subset \mathbb{C}_\infty$ is a finitely generated projective A-submodule of $\mathbb{C}_\infty$ (via ι) such the set $\{\lambda \in \Lambda \mid |\lambda| \leq c\}$ is finite for any $c > 0$. Given such a lattice Λ of rank r, there is a unique power series e_Λ of the form $\sum_{n=0}^\infty a_n z^{q^n}$, $a_n \in \mathbb{C}_\infty$, with infinite radius of convergence, with $a_0 = 1$ and with simple zeros precisely at the points of Λ. The entire function e_Λ is $\mathbb{F}_q$-linear and surjective and hence one has a short exact sequence of $\mathbb{F}_q$-vector spaces

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C}_\infty \xrightarrow{e_\Lambda} \mathbb{C}_\infty \longrightarrow 0.$$

Thus if we let $a \in A$ act by multiplication on the middle and left term, there exists a unique $\mathbb{F}_q$-linear function $\Phi_a(z)$ on the right such that the diagram

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{z \mapsto e_\Lambda(z)} & \mathbb{C}_\infty \\ \big\downarrow {\scriptstyle z\mapsto az} & & \big\downarrow {\scriptstyle z\mapsto \Phi_a(z)} \\ \mathbb{C}_\infty & \xrightarrow{z \mapsto e_\Lambda(z)} & \mathbb{C}_\infty \end{array}$$

commutes. One verifies that Φ_a lies in $\mathbb{C}_\infty[z]_{\mathbb{F}_q}$, and denotes by ϕ_a the corresponding polynomial in $\mathbb{C}_\infty\{\tau\}$. Then $A \to \mathbb{C}_\infty\{\tau\} : a \mapsto \phi_a$ defines a Drinfeld A-module of rank r. It is known that any Drinfeld A-module on $\mathbb{C}_\infty$ of characteristic ι arises in this way.

Suppose now that ϕ is a Drinfeld A-module on L of rank r and generic characteristic $\iota_L : A \hookrightarrow F \hookrightarrow L$. Fix a place v of A, and denote by $\mathfrak{p}_v$ the corresponding maximal ideal of A. Then one defines

$$\phi[v^n] := \{\lambda \in L^{\mathrm{alg}} \bigm| \forall a \in \mathfrak{p}_v^n : \Phi_a(\lambda) = 0\}.$$

The derivative of $\Phi_a(z)$ is the constant $\iota_L(a)$. Since ϕ is of generic characteristic, $\iota_L(a)$ is non-zero, and so the polynomial Φ_a is separable. From this one deduces that $\phi[v^n]$ defines a finite separable Galois extension of L. The *module of v-primary torsion points of* ϕ is

$$\phi[v^\infty] := \bigcup_n \phi[v^n] \subset L^{\mathrm{sep}}.$$

It is stable under the action of G_L and, as an A-module, isomorphic to $(F_v/A_v)^r$, where F_v is the fraction field of A_v. Finally by $\mathrm{Tate}_v(\phi) := \mathrm{Hom}_A(F_v/A_v, \phi[v^\infty])$ one denotes the v-adic Tate module of ϕ.

Example 1. Let $F := \mathbb{F}_q(t)$, $A := \mathbb{F}_q[t]$ and L a finite extension of F. The point ∞ therefore corresponds to the valuation v_∞ which to a quotient f/g of polynomials assigns $\deg g - \deg f$. A Drinfeld A-module $\phi\colon A \to L\{\tau\}$ is uniquely determined by the image of t. If $\iota_L\colon A \hookrightarrow F \hookrightarrow L$ is the characteristic of ϕ and if ϕ is of rank r, then ϕ_t must be of the form

$$t + a_1\tau + \ldots + a_r\tau^r \in L\{\tau\}$$

with $a_r \in L^\times$, and where we identify $t = \iota_L(t)$. Conversely any such polynomial defines a Drinfeld A-module of characteristic ι_L and rank r.

Suppose v is the place corresponding to the maximal ideal (t). Then $\Phi_t(z) = tz + a_1 z^q + \ldots + a_r z^{q^r}$ and

$$\phi[v^\infty] = \{\lambda \in L^{\mathrm{sep}} \bigm| \exists n : \underbrace{\Phi_t \circ \ldots \circ \Phi_t}_{n}(\lambda) = 0\}.$$

To generalize the notion of a Drinfeld A-module to arbitrary schemes over $\mathbb{F}_q$, we now give an (equivalent) alternative definition of a Drinfeld A-module on K. Let $\mathbb{G}_{a,K}$ be the additive group (scheme) on K. By $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K})$ we denote the ring of $\mathbb{F}_q$-linear endomorphisms of the group scheme $\mathbb{G}_{a,K}$. This ring is known to be generated over K by the Frobenius endomorphism τ of $\mathbb{G}_{a,K}$, and thereby isomorphic to $K\{\tau\}$. Thus we may define a Drinfeld A-module as a non-constant homomorphism

$$\phi\colon A \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K}).$$

Since the elements of $a \in A$ act as endomorphisms of the group scheme $\mathbb{G}_{a,K}$, they induce an endomorphism $\partial\phi_a$ on the corresponding Lie algebra. This is called the derivative of ϕ and yields a ring homomorphism

$$\partial\phi\colon A \to \mathrm{End}(\mathrm{Lie}\,\mathbb{G}_{a,K}) \cong \Gamma(\mathrm{Spec}\,K, \mathcal{O}_{\mathrm{Spec}\,K}) = K. \tag{1}$$

This homomorphism is precisely the characteristic of ϕ.

Over an arbitrary $\mathbb{F}_q$-scheme X one defines a Drinfeld A-module as follows: Let $\mathcal{L}$ denote a line bundle on X (i.e., a scheme which is an $\mathbb{G}_a$-bundle on X), and $\mathrm{End}_{\mathbb{F}_q}(\mathcal{L})$ the ring of $\mathbb{F}_q$-linear endomorphisms of the group scheme $\mathcal{L}$ over X. A *Drinfeld A-module of rank r on* $(X, \mathcal{L})$ is a homomorphism

$$\phi\colon A \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{L}),$$

such that for all fields K and all morphisms $\pi : \mathrm{Spec}\,K \to X$ the induced homomorphism $\phi\colon A \to \mathrm{End}_{\mathbb{F}_q}(\pi^*\mathcal{L})$ is a Drinfeld A-module of rank r on K.

In the same way as (1) was constructed, ϕ induces a homomorphism $\partial\phi\colon A \to \Gamma(X, \mathcal{O}_X)$. The corresponding morphism of schemes $\mathrm{char}_\phi\colon X \to \mathrm{Spec}\,A$ is called the characteristic of ϕ. Via the characteristic, X becomes an A-scheme (, i.e., a scheme with a morphism to $\mathrm{Spec}\,A$).

A morphism from a Drinfeld A-module ϕ on $\mathcal{L}$ to ϕ' on $\mathcal{L}'$ is a morphism $\alpha \in \mathrm{Hom}_{\mathbb{F}_q}(\mathcal{L}, \mathcal{L}')$ which is A-equivariant, i.e., such that for all $a \in A$ one has $\alpha\phi_a = \phi'_a\alpha$.

3 Anderson's motives

In the preceding section we encountered an object defined over function fields, namely a Drinfeld A-module, which has properties seemingly similar to those of an elliptic curve. Therefore it seemed natural to look for a category of motives which would naturally contain all Drinfeld A-modules (of generic characteristic). In particular this category should be A-linear, it should allow for constructions from linear algebra such as sums, tensor, exterior and symmetric products, and it should have étale as well as analytic realizations. It was Anderson in his seminal paper [An86] who first realized how to construct such a theory, perhaps motivated by the earlier definition of shtuka due to Drinfeld. Some further details can be found in [vdH03], Chap. 4.

We fix a subfield K of $\mathbb{C}_\infty$ containing F, denote by σ the Frobenius on K relative to $\mathbb{F}_q$, and set $\iota_K\colon A \hookrightarrow F \hookrightarrow K$.

Definition 1. *An* abelian A-motive on K *consists of a pair (M, τ) such that*

1. *M is a finitely generated projective $K \otimes A$-module.*
2. *$\tau\colon M \to M$ is an injective σ-semilinear endomorphism of M, i.e., for all $m \in M$, $x \in K$ and $a \in A$ one has $\tau((x \otimes a)m) = (x^q \otimes a)\tau(m)$.*
3. *The module $M/K\tau(M)$ is of finite length over $K \otimes A$, and annihilated by a power of the maximal ideal generated by $\{\iota(a) \otimes 1 - 1 \otimes a \mid a \in A\}$.*

4. $M \otimes_K K^{\mathrm{perf}}$ is finitely generated over $K^{\mathrm{perf}}\{\tau\}$, where K^{perf} denotes the perfect closure of K.

Using that $K^{\mathrm{perf}}\{\tau\}$ is a left principal ideal domain, it is shown in [An86] that for any abelian A-motive M the module $M \otimes_K K^{\mathrm{perf}}$ is free over $K^{\mathrm{perf}}\{\tau\}$. The rank of $M \otimes K^{\mathrm{perf}}$ over $K^{\mathrm{perf}}\{\tau\}$ is called the *dimension of* M, its rank over $K \otimes A$ is called the *rank of* M.

If K is perfect, condition 3 can be simplified using $M/K\tau(M) = M/\tau(M)$. The maximal ideal in 3 defines a K-rational point of $\mathrm{Spec}(K \otimes A)$.

Definition 2. *A pair (M, τ) which satisfies conditions 1 to 3 only, is called an* A-motive *on K.*

This definition differs significantly from [Go96], Def. 5.4.2, while Definition 1 is the same as in [Go96] and [An86]. One has the following obvious result:

Proposition 1. *If (M, τ) and (M', τ') are abelian A-motives on K, then so is $(M \oplus M', \tau \oplus \tau')$.*

If (M, τ) and (M', τ') are A-motives on K, then so is their tensor product, as well as all tensor, exterior and symmetric powers of (M, τ).

To define the analytic realization of an A-motive, we have to introduce some further notation. The Tate algebra over $\mathbb{C}_\infty$ is defined as

$$\mathbb{C}_\infty\langle t\rangle := \Big\{ \sum a_n t^n \;\Big|\; a_n \in \mathbb{C}_\infty, |a_n| \to 0 \text{ for } n \to \infty \Big\}.$$

Any monomorphism $\mathbb{F}_q[t] \hookrightarrow A$ is finite and flat. Fixing one, any module M underlying some A-motive can be regarded as a (free and finitely generated) module over $K[t]$. We set

$$M\langle t\rangle := M \otimes_{K[t]} \mathbb{C}_\infty\langle t\rangle,$$

and define $M\langle t\rangle^\tau$ as the $\mathbb{F}_q[t]$-module of τ-invariant elements of $M\langle t\rangle$. The module of τ-invariants is a projective (left) A-module and satisfies

$$\mathrm{rank}_A\, M\langle t\rangle^\tau \le \mathrm{rank}_{K\otimes A}\, M. \tag{2}$$

Definition 3. *If equality holds in (2), then (M, τ) is called* uniformizable.

Anderson gave examples of abelian A-motives of dimension greater than one which are not uniformizable. If (M, τ) is uniformizable, we regard the A-module $M\langle t\rangle^\tau$ as its *analytic realization.* It is also shown in [An86], that for any uniformizable motive M of dimension d and rank r, there is a related short exact sequence

$$0 \longrightarrow \Lambda_M \longrightarrow \mathbb{C}_\infty^d \xrightarrow{e_M} \mathbb{C}_\infty^d \longrightarrow 0,$$

where Λ_M is a discrete A-lattice in $\mathbb{C}_\infty^d$ of rank r. Let Ω_A denote the module of Kähler differentials of A over $\mathbb{F}_q$. Then by [An86], § 2, there is the following isomorphism which relates Λ_M with the analytic realization of M:

$$\mathrm{Hom}_A(M\langle t\rangle^\tau, \Omega_A) \cong \Lambda. \tag{3}$$

Following Anderson, [An86], § 2, one has a fully faithful functor from Drinfeld A-modules to uniformizable A-motives of dimension 1: Let ϕ be a Drinfeld A-module on K of generic characteristic ι_K. Via left multiplication by K and right multiplication by ϕ_a for $a \in A$, we regard $M(\phi) := K\{\tau\}$ as a module over $K \otimes A$. Left multiplication by τ defines a σ-semilinear endomorphism $\tau\colon M(\phi) \to M(\phi)$.

Theorem 1. *The assignment $\phi \mapsto (M(\phi), \tau)$ defines a functor which identifies the category of Drinfeld A-modules of rank r with the category of abelian A-motives M of rank r that satisfy $M \cong K\{\tau\}$. Any such A-motive is uniformizable. Moreover if $K = \mathbb{C}_\infty$ and ϕ arises from a lattice Λ, then Λ and $M\langle t\rangle^\tau$ are related via (3).*

To define *étale realizations*, one proceeds as follows: Let $\mathfrak{p}_v$ be the prime ideal of A corresponding to the place v of A. Then

$$M \otimes_{K\otimes A} (K^{\mathrm{sep}} \otimes A/\mathfrak{p}_v^n)$$

is free and finitely generated over $K^{\mathrm{sep}} \otimes A/\mathfrak{p}_v^n$. By condition 3 for a motive, the induced τ-action is in fact bijective. From Lang's theorem, [An86], 1.8.2, one easily deduces that

$$M_{v,n} := \Big(M \otimes_{K\otimes A} (K^{\mathrm{sep}} \otimes A/\mathfrak{p}_v^n)\Big)^\tau$$

is a free $A/\mathfrak{p}_v^n$-module of rank r. Because M is defined over K, the actions of G_K and of τ commute on $M \otimes_{K\otimes A} (K^{\mathrm{sep}} \otimes A/\mathfrak{p}_v^n)$, and so G_K acts $A/\mathfrak{p}_v$-linearly on $M_{v,n}$. The inverse limit $M_{v,\infty} := \varprojlim M_{v,n}$ is thus a free A_v-module of rank r with a continuous linear action of G_K. This we regard as the *v-adic étale realization of M*. It exists independently of the uniformizability of M. One has the following result due to Anderson:

Proposition 2. *Suppose ϕ is a Drinfeld A-module of rank r and generic characteristic, and $(M, \tau) := (M(\phi), \tau)$. Then there is a canonical isomorphism*

$$\mathrm{Hom}_{A_v}(M_{v,\infty}, \varinjlim \mathfrak{p}_v^{-n}\Omega_A/\Omega_A) \cong \phi[v^\infty].$$

4 A cohomological framework

In the previous section, we have seen that Anderson's category of A-motives has a number of very useful properties. It allows constructions from linear algebra, it has realizations as one would expect them from motives, and it contains the category of Drinfeld A-modules. Also pullback along morphisms $Y \to X$ for families of A-motives on a scheme X is easily defined.

Anderson's theory does not, however, provide a cohomological theory of A-motive like objects, which is also desirable and by which we mean the following: On every base scheme X over $\mathbb{F}_q$ we would like to have a category of objects similar to families of A-motives. For any morphism $f\colon Y \to X$, there should be (derived) functors $R^i f_*$ between these categories, and perhaps also other ones such as f^*, $\otimes$, Hom, $Rf_!$, etc.

In [BP04] such a theory is developed in joint work with R. Pink. Much of the material described in the previous section was inspirational for this. The main motivation for the work in [BP04] was to give a cohomological proof of a rationality conjecture by Goss, that had, by analytical methods, previously been established in work of Taguchi and Wan, cf. [TW96]. This is discussed in greater detail in Subsection 5.1.

An alternative construction of such a cohomological theory was recently also described by M. Emerton and M. Kisin in [EK04]. The main reference for the present section is [BP04].

Conventions: Throughout X, Y, etc., will be noetherian schemes over $\mathbb{F}_q$. By σ_X or simply σ we denote the absolute Frobenius endomorphism of X relative to $\mathbb{F}_q$. We fix a morphism of schemes $f\colon Y \to X$.

The symbol B (or B') will always denote an $\mathbb{F}_q$-algebra which arises as a localization of an $\mathbb{F}_q$-algebra of finite type. Typically B will be A, or $A/\mathfrak{a}$ for some ideal of A, or the fraction field F of A.

Whenever tensor or fiber products are formed over $\mathbb{F}_q$, the subscript $\mathbb{F}_q$ at $\otimes$ and $\times$ will be omitted.

By $\mathrm{pr}_1\colon X \times \operatorname{Spec} B \to X$ the projection onto the first factor is denoted.

4.1 τ-sheaves

Definition 4. *A* τ-sheaf over B *on a scheme X is a pair $\underline{\mathcal{F}} := (\mathcal{F}, \tau_{\mathcal{F}})$ consisting of a coherent sheaf $\mathcal{F}$ on $X \times \operatorname{Spec} B$ and an $\mathcal{O}_{X\times \operatorname{Spec} B}$-linear homomorphism*

$$(\sigma \times \mathrm{id})^* \mathcal{F} \xrightarrow{\tau} \mathcal{F}.$$

A homomorphism *of τ-sheaves $\underline{\mathcal{F}} \to \underline{\mathcal{G}}$ over B on X is a homomorphism of the underlying sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ which is compatible with the action of τ.*

We often simply speak of *τ-sheaves on X*. The sheaf underlying a τ-sheaf $\underline{\mathcal{F}}$ will always be denoted by $\mathcal{F}$. When the need arises to indicate on which sheaf τ acts, we write $\tau = \tau_{\mathcal{F}}$.

The category of all τ-sheaves over A on X is denoted by $\mathbf{Coh}_\tau(X, A)$. It is an abelian A-linear category, and all constructions like kernel, cokernel, etc. are the usual ones on the underlying coherent sheaves, with the respective τ added by functoriality.

In this survey we focus on coherent objects. To alleviate our notation, we deviate from the terminology in [BP04]. What is called a τ-sheaf here is a coherent τ-sheaf in [BP04].

On any affine open $\operatorname{Spec} R \subset X$ a τ-sheaf over B is given by a finitely generated $R \otimes B$-module M together with an $R \otimes B$-linear homomorphism $R^\sigma \otimes_R M \to M$. The latter homomorphism corresponds bijectively to a $\sigma \otimes \mathrm{id}$-linear morphism $\tau\colon M \to M$. The pair (M, τ) is also called a *τ-module*.

Example 2. Due to properties 1 and 2 in the definition of a motive, any A-motive on K is a τ-module, and thus yields a τ-sheaf over A on $\operatorname{Spec} K$. However the notion of τ-sheaf is less restrictive than that of A-motive, since we impose no local freeness conditions, nor conditions on the kernel or cokernel of τ.

4.2 Examples

We now describe some further examples of τ-sheaves. All but the first and last of these correspond to families of A-motives (of fixed rank).

Most of the examples are *locally free τ-sheaves*; by this we mean τ-sheaves whose underlying sheaf is locally free. The *rank* of a locally free τ-sheaf is that of its underlying sheaf. Locally free τ-sheaves had been considered already in [TW96], where they were called ϕ-sheaves.

I. Any (coherent) sheaf $\mathcal{F}$ on $X \times \operatorname{Spec} B$ can be made into a τ-sheaf by setting $\tau = 0$. As we will see shortly, these τ-sheaves are not interesting to us.

II. The unit τ-sheaf, denoted by $\underline{\mathbb{1}}_{X,B}$, is the free τ-sheaf defined by the pair consisting of the sheaf $\mathcal{O}_{X \times \operatorname{Spec} B}$ together with the isomorphism

$$\tau\colon (\sigma \times \mathrm{id})^* \mathcal{O}_{X \times \operatorname{Spec} B} \longrightarrow \mathcal{O}_{X \times \operatorname{Spec} B},$$

which via adjunction arises from $\sigma \otimes \mathrm{id}\colon \mathcal{O}_{X \times \operatorname{Spec} B} \longrightarrow (\sigma \times \mathrm{id})_* \mathcal{O}_{X \times \operatorname{Spec} B}$.

III. The construction $\phi \mapsto M(\phi)$ from Drinfeld A-modules to A-motives generalizes in an obvious way to a functor $(\mathcal{L}, \phi) \mapsto \underline{\mathcal{M}}(\phi)$ from Drinfeld A-modules over a general base X to τ-sheaves over A on X. The rank of the Drinfeld module becomes the rank of the locally free sheaf underlying $\underline{\mathcal{M}}(\phi)$.

Similarly any elliptic sheaf on X gives rise to a τ-sheaf over A on X, by restricting it to $X \times \operatorname{Spec} A$. For details on this, we refer to the excellent article of Blum and Stuhler, cf. [BS97].

IV. As we have seen earlier, one can define Drinfeld A-modules over a general base. So it is natural to consider corresponding moduli problems. As in the case of elliptic curves (or abelian varieties) these are not rigid, unless one introduces some level structures. To define such, we fix a proper non-zero ideal $\mathfrak{n} \subset A$. Then for any Drinfeld A-module $\phi\colon A \to \operatorname{End}_{\mathbb{F}_q}(\mathcal{L})$ of rank r over a scheme X, one defines the subscheme

$$\mathcal{L}_\phi[\mathfrak{n}] := \bigcap_{a \in \mathfrak{n}} \operatorname{Ker}(\mathcal{L} \xrightarrow{\phi_a} \mathcal{L}) \subset \mathcal{L}.$$

It carries an action of $A/\mathfrak{n}$ and is finite flat over X. Its degree over X is the cardinality of $(A/\mathfrak{n})^r$. If the image of $\mathrm{char}_\phi : X \to \operatorname{Spec} A$ is disjoint from

$\operatorname{Spec} A/\mathfrak{n}$, we say that *the characteristic of* ϕ *is prime to* $\mathfrak{n}$. In this case $\mathcal{L}_\phi[\mathfrak{n}]$ is moreover étale and Galois over X.

For a finite discrete group G we denote by G_X the corresponding constant group scheme on X.

Definition 5. *A* naive level $\mathfrak{n}$-structure on ϕ *is an isomorphism*

$$\psi\colon (A/\mathfrak{n})^r_X \xrightarrow{\cong} \mathcal{L}_\phi[\mathfrak{n}].$$

A naive level $\mathfrak{n}$-structure can only exist if the characteristic of ϕ is prime to $\mathfrak{n}$. In that case it always exists over a finite Galois covering $Y \to X$.

Let $A(\mathfrak{n})$ denote the ring of rational functions in F regular outside ∞ and the primes dividing $\mathfrak{n}$. Then for any fixed $r \in \mathbb{N}$ one may consider the moduli functor $M^r(\mathfrak{n})$ which to an $A(\mathfrak{n})$-scheme X assigns the set of triples $(\mathcal{L}, \phi, \psi)$ (up to isomorphism), where

1. $\phi\colon A \to \operatorname{End}_{\mathbb{F}_q}(\mathcal{L})$ is a Drinfeld A-module of rank r, and
2. $\psi\colon (A/\mathfrak{n})^r_X \xrightarrow{\cong} \mathcal{L}_\phi[\mathfrak{n}]$ is a naive level $\mathfrak{n}$-structure,

such that the composite of the structure morphism $X \to \operatorname{Spec} A(\mathfrak{n})$ with the canonical open immersion $\operatorname{Spec} A(\mathfrak{n}) \hookrightarrow \operatorname{Spec} A$ is equal to char_ϕ.

Theorem 2 ([Dr76]). *The moduli problem* $M^r(\mathfrak{n})$ *is representable by an affine (noetherian)* $A(\mathfrak{n})$*-scheme* $\mathfrak{Y}^r(\mathfrak{n})$*. The line bundle* $\mathcal{L}^r(\mathfrak{n})$ *in its universal triple* $(\mathcal{L}^r(\mathfrak{n}), \phi^r(\mathfrak{n}), \psi^r(\mathfrak{n}))$ *on* $\mathfrak{Y}^r(\mathfrak{n})$ *is isomorphic to* $\mathbb{G}_{a,\mathfrak{Y}^r(\mathfrak{n})}$*. The structure morphism* $\mathfrak{Y}^r(\mathfrak{n}) \to \operatorname{Spec} A(\mathfrak{n})$ *is smooth of relative dimension* $r-1$.

For later use, we record the following:

Proposition 3. *The* τ*-sheaf* $\underline{\mathcal{M}}^r(\mathfrak{n}) := \underline{\mathcal{M}}(\phi^r(\mathfrak{n}))$ *is locally free of rank* r.

V. One can in fact define moduli spaces of more general types of A-motives (which carry some polarization and level structure). The corresponding universal A-motive, then again yields interesting τ-sheaves. The investigation of some of these moduli problems is ongoing work by U. Hartl and, independently, L. Taelman.

VI. An important notion to study the reduction of abelian varieties is their Néron model. For τ-sheaves, Gardeyn in [Ga03] introduced and investigated the following substitute:

Definition 6. *Suppose* $j : U \hookrightarrow X$ *a dense open immersion. A* τ*-sheaf* $\underline{\mathcal{G}} \in \mathbf{Coh}_\tau(X, A)$ *is called a* model of $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U, A)$ with respect to j *if* $j^*\underline{\mathcal{G}} \cong \underline{\mathcal{F}}$. *A model* $\underline{\mathcal{G}}$ *is called a* maximal model of $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U, A)$ with respect to j *if for all* $\underline{\mathcal{H}} \in \mathbf{Coh}_\tau(X, A)$ *the canonical homomorphism*

$$\operatorname{Hom}_{\mathbf{Coh}_\tau(X,A)}(\underline{\mathcal{H}}, \underline{\mathcal{G}}) \longrightarrow \operatorname{Hom}_{\mathbf{Coh}_\tau(U,B)}(\underline{\mathcal{H}}_{|U \times \operatorname{Spec} A}, \underline{\mathcal{F}}) \tag{4}$$

is an isomorphism. We write $j_\#\underline{\mathcal{F}} := \underline{\mathcal{G}}$.

If it exists, a maximal model is always unique up to unique isomorphism. There always exists a direct limit of τ-sheaves $\underline{\mathcal{G}}$ which satisfies (4). The crucial requirement is that $\mathcal{G}$ be coherent.

As to be expected, if $\phi\colon A \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{L})$ is a Drinfeld A-module on X, and $j\colon U \hookrightarrow X$ a dense open immersion then $j_{\#}\underline{\mathcal{M}}(j^*\phi) \cong j_{\#}(j^*\underline{\mathcal{M}}(\phi)) \cong \underline{\mathcal{M}}(\phi)$.

Theorem 3 ([Ga03]). *Suppose X is a smooth projective curve over $\mathbb{F}_q$, and $j\colon U \hookrightarrow X$ is open and dense. Then for any locally free $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U, A)$ with $\tau_{\mathcal{F}}$ injective, a maximal model $j_{\#}\underline{\mathcal{F}}$ exists. It is again locally free.*

Remark 1. Suppose now that $\phi\colon A \to K\{\tau\}$ is a Drinfeld A-module of rank r over a complete discretely valued field K with ring of integers V, residue field k and $j : \mathrm{Spec}\, K \hookrightarrow \mathrm{Spec}\, V$. Suppose also that ϕ has reduction of rank $0 < r' < r$ over k. (The latter means that over K the Drinfeld-module ϕ is isomorphic to some Drinfeld-module ϕ' whose image lies in $V\{\tau\}$, and such that the reduction of ϕ' to k is a Drinfeld-module of rank r' over k.)

Drinfeld, cf. [Dr76], § 7, has shown that in this situation, after possibly passing to a finite extension of K, there exist a Drinfeld module ϕ' of rank r' over K, a discrete A-sublattice $\Lambda \subset K$ (where A acts on K via ϕ'), and an "analytic" A-homomorphism $e_\Lambda\colon K \to K$, where on the left A acts via ϕ' and on the right via ϕ, such that there is a short exact sequence of A-modules

$$0 \longrightarrow \Lambda \longrightarrow K^{\mathrm{alg}} \xrightarrow{e_\Lambda} K^{\mathrm{alg}} \longrightarrow 0.$$

The transition from Drinfeld-modules to τ-sheaves is contravariant, so the morphism e_Λ turns into an "analytic" morphism between A-motives $M^{\mathrm{an}}(\phi) \longrightarrow M^{\mathrm{an}}(\phi')$ over K. It is again surjective, and its kernel is the analytification of $\underline{\mathbb{1}}^{r-r'}_{\mathrm{Spec}\, K,A}$. So there is a short exact sequence

$$0 \longrightarrow (\underline{\mathbb{1}}^{r-r'}_{\mathrm{Spec}\, K,A})^{\mathrm{an}} \longrightarrow M^{\mathrm{an}}(\phi) \longrightarrow M^{\mathrm{an}}(\phi') \longrightarrow 0.$$

Gardeyn shows that: This short exact sequence can be extended to a left exact sequence over a formal scheme attached to $V \otimes A$. The left two terms of the extended sequence arise via analytification from

$$0 \longrightarrow \underline{\mathbb{1}}^{r-r'}_{\mathrm{Spec}\, V,A} \longrightarrow j_{\#}\underline{\mathcal{M}}(\phi).$$

The induced morphism $\underline{\mathbb{1}}^{r-r'}_{\mathrm{Spec}\, k,A} \longrightarrow i^*j_{\#}\underline{\mathcal{M}}(\phi)$ on the special fiber over $\mathrm{Spec}\, k$ is injective and some iterate of τ vanishes on the cokernel.

The upshot of this longwinded explanation is that in the present situation the maximal extension and the Drinfeld module ϕ' shed light on, so to speak, opposite aspects of the bad reduction situation given. Also while ϕ' can be obtained from ϕ only via an analytic morphism, the relation between ϕ and $j_{\#}\underline{\mathcal{M}}(\phi)$ is algebraic. For more on this interesting topic, we refer to [Ga03]. In this survey maximal extensions will reappear in Remark 2 and Subsection 6.8.

4.3 Crystals

There is a large class of homomorphism in $\mathbf{Coh}_\tau(X,B)$ which one would like to regard as isomorphisms. Categorically, the correct way to deal with this is to localize at this class, cf. [We94], § 10.3. The reason in [BP04] to pass to the localized theory was a practical one: only there we were able to construct an 'extension by zero functor' as sketched below Theorem 4. First we describe the localization procedure:

For a τ-sheaf $\underline{\mathcal{F}}$, we define the iterates τ^n of τ by setting inductively $\tau^0 := \mathrm{id}$ and $\tau^{n+1} := \tau \circ (\sigma \times \mathrm{id})^*\tau^n : (\sigma^{n+1} \times \mathrm{id})^*\mathcal{F} \longrightarrow \mathcal{F}$. A τ-sheaf $\underline{\mathcal{F}}$ is called *nilpotent* if and only if $\tau^n_{\mathcal{F}}$ vanishes for some $n > 0$. A corresponding notion for homomorphisms is:

Definition 7. *A morphism of τ-sheaves is called a* nil-isomorphism *if and only if both its kernel and cokernel are nilpotent.*

It is shown in [BP04], Chap. 2, that the nil-isomorphisms in $\mathbf{Coh}_\tau(X,B)$ form a saturated multiplicative system, and so one defines:

Definition 8. *The category $\mathbf{Crys}(X,B)$ of B-*crystals *on X is the localization of $\mathbf{Coh}_\tau(X,B)$ with respect to nil-isomorphisms.*

The category $\mathbf{Crys}(X,B)$ is again a B-linear abelian category with the induced notions of kernel, image, cokernel and coimage. Its objects are the same as those in $\mathbf{Coh}_\tau(X,A)$. However the homomorphisms are different. Any homomorphism $\underline{\mathcal{F}} \longrightarrow \underline{\mathcal{G}}$ in $\mathbf{Crys}(X,A)$ is represented by a diagram $\underline{\mathcal{F}} \longleftarrow \underline{\mathcal{H}} \longrightarrow \underline{\mathcal{G}}$ in $\mathbf{Coh}_\tau(X,A)$, for some $\mathcal{H} \in \mathbf{Coh}_\tau(X,A)$, and where the homomorphism $\underline{\mathcal{H}} \longrightarrow \underline{\mathcal{F}}$ is a nil-isomorphism.

Let us conclude this subsection with the simplest functor that is based on nilpotency. For a τ-sheaf $\underline{\mathcal{F}}$ on X over B, define

$$\underline{\mathcal{F}}^\tau := \Gamma(X \times \operatorname{Spec} B, \underline{\mathcal{F}})^\tau \tag{5}$$

as the B-module of global τ-invariant sections of $\underline{\mathcal{F}}$. The following result is an immediate consequence of Definition 8.

Proposition 4. *The functor $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^\tau$ from $\mathbf{Coh}_\tau(X,B)$ to B-modules is invariant under nil-isomorphisms. It therefore passes to a functor on the category $\mathbf{Crys}(X,B)$, which is again denoted $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}^\tau$.*

4.4 Functors

On τ-sheaves, we now indicate the construction of four functors:

(a) $f^* : \mathbf{Coh}_\tau(X,B) \longrightarrow \mathbf{Coh}_\tau(Y,B)$ (*pullback*):

For $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X,B)$, we define $f^*\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(Y,B)$ as the pair consisting of the coherent sheaf $(f\times \mathrm{id})^*\mathcal{F}$ on $Y \times \operatorname{Spec} B$ and the $\sigma\times\mathrm{id}$-linear endomorphism on $(f\times\mathrm{id})^*\mathcal{F}$ induced by functoriality from τ. This defines a B-linear functor f^*.

In an analogous way we define B-linear (bi-)functors

(b) $R^if_* : \mathbf{Coh}_\tau(Y,B) \longrightarrow \mathbf{Coh}_\tau(X,B)$ (*push-forward*) if $f: Y \to X$ is proper, and $i \geq 0$.

(c) $\_\_ \otimes \_\_ : \mathbf{Coh}_\tau(X,B) \times \mathbf{Coh}_\tau(X,B) \longrightarrow \mathbf{Coh}_\tau(X,B)$ (*tensor product*).

(d) $\_\_ \otimes_B B' : \mathbf{Coh}_\tau(X,B) \longrightarrow \mathbf{Coh}_\tau(X,B')$ (*change of coefficients*) for any homomorphism $B \to B'$.

Despite the notation, it is at this point not at all clear that the R^if_* are derived functors.

In the same way as the tensor product is defined, following [Ha77], Ex. II.5.16, for the construction on the underlying sheaf, one also obtains higher tensor, symmetric and exterior powers of a τ-sheaf $\underline{\mathcal{F}}$. We denote these by $\otimes^n\underline{\mathcal{F}}$, $\operatorname{Sym}^n \underline{\mathcal{F}}$ and $\bigwedge^n \underline{\mathcal{F}}$. The latter two are quotients of $\otimes^n\underline{\mathcal{F}}$.

The functors defined in (a)–(d) as well as $\otimes^n$, Sym^n and $\bigwedge^n$ all preserve nil-isomorphisms. Hence they pass to functors between the corresponding categories of crystals:

Theorem 4. *The functors f^*, R^if_*, $\otimes$ and $\otimes_B B'$ on τ-sheaves induce B-linear functors*

(a) $f^* : \mathbf{Crys}(X,B) \to \mathbf{Crys}(Y,B)$.
(b) $R^if_* : \mathbf{Crys}(Y,B) \to \mathbf{Crys}(X,B)$ *for $i \geq 0$, if f is proper.*
(c) $\_\_ \otimes \_\_ : \mathbf{Crys}(X,B) \times \mathbf{Crys}(X,B) \to \mathbf{Crys}(X,B)$.
(d) $\_\_ \otimes_B B' : \mathbf{Crys}(X,B) \to \mathbf{Crys}(X,B')$.

If f is proper, then f_* and f^* form an adjoint functor pair on crystals. Moreover $\otimes^n$, Sym^n and $\bigwedge^n$ induce functors on $\mathbf{Crys}(X,B)$.

Now we come to a main point, namely the construction of an extension by zero in the theory of crystals. Such a functor is not present on τ-sheaves. We shall see how localization affords this functor on crystals.

To explain the construction, let $j: U \hookrightarrow X$ be an open embedding with $i: Z \hookrightarrow X$ a closed complement, and denote by $\mathcal{I}_0 \subset \mathcal{O}_X$ the ideal sheaf of Z. Then $\mathcal{I} := \mathrm{pr}_1^*\mathcal{I}_0$ is the ideal sheaf for $Z \times \operatorname{Spec} B \subset X \times \operatorname{Spec} B$. We wish to extend any τ-sheaf $\underline{\mathcal{F}}$ on U "by zero" to X.

By local considerations on X, one can alway construct some coherent extension $\widetilde{\mathcal{F}}$ on $X \times \operatorname{Spec} B$ of $\mathcal{F}$. Such an extension is by no means unique, since any sheaf $\mathcal{I}^m\widetilde{\mathcal{F}}$, $m \in \mathbb{N}$, is also an extension of $\mathcal{F}$. To extend τ, we observe that for some $\widetilde{m} \in \mathbb{N}$ it extends to a homomorphism $(\sigma\times\mathrm{id})^*\mathcal{I}^{\widetilde{m}}\widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$. The identity $(\sigma\times\mathrm{id})^*\mathcal{I} = \mathcal{I}^q$ inside $\mathcal{O}_{X\times \operatorname{Spec} B}$ implies that for any $m \gg 0$ one has an extension

$$\tau_m\colon (\sigma\times \mathrm{id})^*\mathcal{I}^m\widetilde{\mathcal{F}} \to \mathcal{I}^{m+1}\widetilde{\mathcal{F}} \overset{(*)}{\subset} \mathcal{I}^m\widetilde{\mathcal{F}}$$

of τ. By its construction, the pair $\underline{\widetilde{\mathcal{F}}}_m := (\mathcal{I}^m\widetilde{\mathcal{F}}, \tau_m)$ is a τ-sheaf on X which extends $\underline{\mathcal{F}}$. The inclusion $(*)$ implies that $i^*\underline{\widetilde{\mathcal{F}}}_m$ is nilpotent – in fact the induced τ is zero –, and so $\underline{\widetilde{\mathcal{F}}}_m$ has the properties of an extension by zero. Except the assignment $\underline{\mathcal{F}} \mapsto \underline{\widetilde{\mathcal{F}}}_m$ is in no way functorial since there are many possible choices. The key observation is that all such pairs are nil-isomorphic. Passing to crystals yields the following result:

Theorem 5. *There is an exact B-linear functor*

$$j_!\colon \mathbf{Crys}(U,B) \to \mathbf{Crys}(X,B) : \underline{\mathcal{F}} \mapsto j_!\underline{\mathcal{F}},$$

uniquely characterized by the properties $j^*j_! = \mathrm{id}_{\mathbf{Crys}(U,B)}$ and $i^*j_! = 0$.

Having an extension by zero, it is well-known how to define cohomology with compact supports:

Definition 9 (Cohomology with compact support). *Say f is compactifiable, i.e., $f = \bar{f}\,\bar{j}$ for some open immersion $\bar{j} : Y \hookrightarrow \overline{Y}$ and some proper morphism $\bar{f}\colon \overline{Y} \to X$. Then one defines*

$$R^i f_! := R^i \bar{f}_* \circ \bar{j}_!\colon \mathbf{Crys}(Y,B) \longrightarrow \mathbf{Crys}(X,B).$$

Standard arguments show that the definition is independent of the chosen factorization, e.g. [Mi80], Chap. VI, §3. Furthermore, due to a result of Nagata, any morphism $f : Y \to X$ between schemes of finite type over $\mathbb{F}_q$ is compactifiable, and so in this situation the $R^i f_!$ exist, cf. [Lü93] or [CoDe].

4.5 Sheaf-theoretic properties

Since $\mathbf{Crys}(X,B)$ is an abelian category, one has the notion of exactness in short sequences. There is also a good notion of *stalk* at a point $x \in X$, where $i_x : x \hookrightarrow X$ denotes the corresponding immersion: The *stalk of a crystal $\underline{\mathcal{F}}$ (on X) at* x is defined as $i_x^*\underline{\mathcal{F}}$. The sheaf underlying the stalk $i_x^*\underline{\mathcal{F}}$ is in general *not* the stalk at x of the sheaf $\mathcal{F}$ underlying $\underline{\mathcal{F}}$. The following result justifies the definition of $i_x^*\underline{\mathcal{F}}$:

Theorem 6.
1. *For any morphism $f\colon Y \to X$, pullback along f is an exact functor on crystals.*
2. *A sequence of crystals is exact if and only if it is exact at all stalks.*
3. *The support of a crystal $\underline{\mathcal{F}}$, i.e., the set of $x \in X$ for which $i_x^*\underline{\mathcal{F}}$ is non-zero, is constructible.*

We note that part 1 is established by showing that the higher right derived functors of f^* on τ-sheaves are all nilpotent.

Moreover crystals enjoy a rigidity property that is not shared by τ-sheaves, but reminiscent of properties of étale sheaves:

Theorem 7. *If f is finite radicial and surjective, the functors*

$$\mathbf{Crys}(X,B) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Crys}(Y,B)$$

are mutually quasi-inverse equivalences of categories.

In particular the closed immersion $f : X_{\mathrm{red}} \hookrightarrow X$ yields an isomorphism $\mathbf{Crys}(X_{\mathrm{red}}, B) \cong \mathbf{Crys}(X, B)$. It is this rigidity property which motivates the name "crystal": crystals extend in a unique way under infinitesimal extensions.

4.6 Derived categories and functors

A major part of [BP04] is to extend the functors (a)–(d) and extension by zero to derived functors between suitable derived categories of crystals. This yields derived functors f^*, $\otimes$, $Rf_!$, and change of coefficients.

There are various good reasons to do so. For instance only there one can properly understand derived functors such as $Rf_!$. It can be shown that the ad hoc defined functors $R^i f_*$ for proper f are indeed i-th cohomologies of a right derived functor Rf_*. Moreover derived categories is the correct setting to discuss the theory of L-functions needed in Section 5.

The objects introduced so far do not suffice to define derived functors. The reader may recall that to properly define the cohomological functors on coherent sheaves, one needs the ambient larger category of quasi-coherent sheaves. Only there one disposes of Čech resolutions and resolutions by injectives. Similarly, in [BP04], two auxiliary categories of τ-sheaves are introduced, namely those of quasi-coherent τ-sheaves, and of inductive limits of coherent τ-sheaves. In the presence of the endomorphism τ, these two and $\mathbf{Crys}(X, B)$ are pairwise distinct. It is in fact an important result of [BP04], that the corresponding derived categories of bounded complexes with coherent cohomology are all equivalent. Having good comparison results of the categories, the treatment of the derived functors follows the usual path.

4.7 Flatness

An important prerequisite to discussing L-functions of crystals in Section 5 is flatness. Only to flat B-crystals we can hope to attach an L-function which takes values in $1 + tB[[t]]$. Since flatness can only be fully understood in derived categories, which we mainly avoid in this survey, we also introduce the notion of crystal of *pullback type*, which is less natural, but technically easier to handle.

Definition 10. *A crystal $\underline{\mathcal{F}}$ is flat if the functor $\underline{\mathcal{F}} \otimes \_\_ : \mathbf{Crys}(X, B) \to \mathbf{Crys}(X, B)$ is exact.*

A crystal is called *(locally) free*, if it may be represented by a (locally) free τ-sheaf. A locally free τ-sheaf is acyclic for $\otimes$ on τ-sheaves, and so it represents a flat crystal. There are other flat crystals which are easy to describe:

A τ-sheaf $\underline{\mathcal{F}} = (\mathcal{F}, \tau)$ on X is of *pullback type*, if there exists a coherent sheaf $\mathcal{F}_0$ on X such that $\mathcal{F} \cong \mathrm{pr}_1^* \mathcal{F}_0$. For an affine scheme $X = \operatorname{Spec} R$, a τ-sheaf is of pullback type if its underlying sheaf is of the form $M_0 \otimes B$ for some finitely generated R-module M_0. A crystal on X is called of *pullback type*, if it can be represented by a τ-sheaf of pullback type. This notion derives its importance from the following proposition:

Proposition 5. *Any crystal of pullback type is flat.*

Being of pullback type is preserved under all the functors defined so far, i.e., under pullback, tensor product (of two crystals of pullback type), change of coefficients, the functors $R^i f_!$, and $\otimes^n$, Sym^n and $\bigwedge^n$.

Let us briefly explain the first assertion of the proposition: Since by Theorem 6 exactness can be verified on stalks, a crystal is flat if and only if all its stalks are flat. From the definition of stalk for a crystal, given above Theorem 6, it follows that if the crystal is represented by a τ-sheaf of pullback type, then its stalk at any point x of X is represented by a τ-sheaf of pullback type over A on $\operatorname{Spec} k_x$. Over the field k_x any module is flat, and hence the pullback of such a module to $k_x \otimes A$ is flat as well. This shows that all the stalks of a crystal of pullback type are flat, which we needed to verify.

Similarly, one has the following important properties of flatness:

Theorem 8.
1. *Flatness of a crystal is preserved under pullback, tensor product (of two flat crystals), change of coefficients and extension by zero.*
2. *If f is compactifiable and $\underline{\mathcal{F}}^\bullet$ is a bounded complex of flat crystals on Y, then $Rf_!\underline{\mathcal{F}}^\bullet$ is represented by a bounded complex of flat crystals.*
3. *A crystal is flat if and only if all its stalks are flat.*

We observed that any locally free crystal is flat. It is in fact not a simple matter to understand precisely when, or in what sense a flat crystal may be represented by a locally free τ-sheaf. This is relevant to us since we want to attach to a flat crystal and a point $x \in X$ with finite residue field an L-function at x. Therefore it would be desirable that its stalk at x have a representing τ-sheaf whose underlying sheaf is free and finitely generated over A. Unfortunately this is not true in general. However we have the following important special case:

Theorem 9. *Suppose that $x = \operatorname{Spec} k_x$ for some finite field extension k_x of $\mathbb{F}_q$, that A is artinian and that $\underline{\mathcal{F}}$ is an A-crystal on x. Then:*

1. *The crystal $\underline{\mathcal{F}}$ has a representing τ-sheaf whose endomorphism τ is an isomorphism.*
2. *The representative from 1 is unique up to unique isomorphism; we write $\underline{\mathcal{F}}_{\mathrm{ss}}$ for it and call it the* semisimple part *of $\underline{\mathcal{F}}$.*

3. *The assignment* $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\mathrm{ss}}$ *is functorial.*
4. *If* $\underline{\mathcal{F}}$ *is flat, then the module underlying* $\underline{\mathcal{F}}_{\mathrm{ss}}$ *is free over* $k_x \otimes A$.

For later use, we also record the following:

Proposition 6. *Suppose that* X *is the spectrum of a field, and* B *is regular of dimension* ≤ 1 *or finite. Then every flat* B*-crystal on* X *can be represented by a locally free* τ*-sheaf on which* τ *is injective.*

If $\underline{\mathcal{F}}$ is represented by a torsion free τ-sheaf, then the proof in [BP04] shows that $\mathrm{Im}(\tau_{\mathcal{F}}^n)$ has the asserted property for $n \gg 0$.

The most general representability result for flat crystals, shown in [BP04], is that for any flat $\underline{\mathcal{F}}$ and any reduced scheme X there is a finite cover by locally closed regular subschemes X_i, so that on each X_i some 'j-th iterate' of $\underline{\mathcal{F}}$ is representable by a 'τ^j-sheaf' whose underlying sheaf is locally free. Since we will not need this, we will not give the details. Note however that this is reminiscent of the definition of constructibility of étale sheaves.

4.8 A test case

Throughout this section we assume that B is finite (and an $\mathbb{F}_q$-algebra). Let $\mathbf{\acute{E}t}(X,B)$ be the category of étale sheaves of B-modules and $\mathbf{\acute{E}t}_c(X,B)$ its full subcategory of constructible sheaves. By $\mathrm{pr}_1 : X \times \operatorname{Spec} B \to X$ we denote the projection onto the first factor.

It was long known that for such B there is a correspondence between τ-sheaves on which τ is an isomorphism and lisse étale constructible sheaves of B-modules, cf. for instance [Ka73], Thm. 4.1.1, which we recall in Theorem 10 below. In [BP04], this correspondence was extended to an equivalence of categories with on the one hand $\mathbf{Crys}(X,B)$ and on the other $\mathbf{\acute{E}t}_c(X,B)$. In this section we describe the functor which provides this equivalence.

We would like to remark that - under the name of 'Riemann-Hilbert correspondence' - M. Emerton and M. Kisin have, for regular X, constructed another equivalence of categories between on the one hand $\mathbf{\acute{E}t}_c(X,B)$ and on the other again a category whose objects carry a σ-semi-linear operation, cf. [EK04]. It turns out, and this is currently under investigation by M. Blickle and the author, that their category is dual to the category of crystals, with the duality given by the duality of sheaves as described by Hartshorne in [Ha66]. Emerton and Kisin have extended their correspondence to formal $\mathbb{Z}_p$-coefficients. It would be interesting to see, whether in some way one can extend the concept of τ-sheaf to incorporate $\mathbb{Z}/(p^n)$-coefficients.

Let $\underline{\mathcal{F}}$ be a τ-sheaf over B on X. To any étale morphism $u : U \to X$ we assign the B-module $(u^*\underline{\mathcal{F}})^\tau$ of τ-invariants of $u^*\underline{\mathcal{F}}$. This construction is functorial in u and hence defines a sheaf of B-modules on the small étale site over X, which we denote by $\underline{\mathcal{F}}_{\acute{e}t}$.

The construction is also functorial in $\underline{\mathcal{F}}$, that is, to any homomorphism $\phi\colon \underline{\mathcal{F}} \to \underline{\mathcal{G}}$ it associates a homomorphism $\phi_{\text{ét}}\colon \underline{\mathcal{F}}_{\text{ét}} \to \underline{\mathcal{G}}_{\text{ét}}$. Therefore it defines an B-linear functor

$$\epsilon\colon \mathbf{Coh}_\tau(X,B) \to \acute{\mathbf{E}}\mathbf{t}(X,B) : \underline{\mathcal{F}} \mapsto \epsilon(\underline{\mathcal{F}}) := \underline{\mathcal{F}}_{\text{ét}}. \tag{6}$$

Let us mention some obvious consequences of the definition of ϵ. For any τ-sheaf $\underline{\mathcal{F}}$ and étale $u\colon U \to X$, one has a left exact sequence

$$0 \longrightarrow \underline{\mathcal{F}}_{\text{ét}}(U) \longrightarrow (u^*\mathcal{F})(U \times \operatorname{Spec} B) \xrightarrow{1-\tau} (u^*\mathcal{F})(U \times \operatorname{Spec} B), \tag{7}$$

which induces a short exact sequence of étale sheaves. In the particular case $\underline{\mathcal{F}} = \underline{\mathbb{1}}_{X,\mathbb{F}_q}$, sequence (7) specializes to the usual Artin-Schreier sequence

$$0 \longrightarrow (\underline{\mathbb{1}}_{X,B})_{\text{ét}} \longrightarrow \mathcal{O}_X \xrightarrow{1-\tau} \mathcal{O}_X.$$

Thus $(\underline{\mathbb{1}}_{X,B})_{\text{ét}}$ is the constant étale sheaf with fiber B.

As another example, suppose $(M(\phi),\tau)$ is the A-motive on K attached to a Drinfeld A-module on K. Then we may apply the functor ϵ to $(M(\phi),\tau)\otimes_A A/\mathfrak{n}$ for any non-zero ideal $\mathfrak{n}$ of A. Essentially by specializing Proposition 2 to finite levels, one obtains the isomorphism

$$\phi[\mathfrak{n}] \cong \operatorname{Hom}_{A/\mathfrak{n}}(\epsilon((M(\phi),\tau)\otimes_A A/\mathfrak{n})(K^{\text{sep}}), \mathfrak{n}\Omega_A/\Omega_A).$$

In particular, ϵ can be used to define étale realizations of A-crystals.

Generalizing Artin-Schreier theory, the following result is proved by Katz in [Ka73], Thm. 4.1.1:

Theorem 10. *For a normal domain X the functor $\underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}$ defines an equivalence between the categories $\{\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X,B) : \tau_{\mathcal{F}}$ is an isomorphism$\}$ and $\{\mathsf{F} \in \acute{\mathbf{E}}\mathbf{t}_c(X,B) : \mathsf{F}$ is lisse$\}$.*

Since for nilpotent τ-sheaves $\underline{\mathcal{F}}$ one has $\underline{\mathcal{F}}_{\text{ét}} = 0$, one easily deduces that ϵ passes to a functor $\mathbf{Crys}(X,B) \to \acute{\mathbf{E}}\mathbf{t}(X,B)$. Using this, [BP04], Chap. 9, refines Theorem 10 to:

Theorem 11. *The functor $\mathbf{Crys}(X,B) \to \acute{\mathbf{E}}\mathbf{t}(X,B) : \underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\text{ét}}$ takes its image in $\acute{\mathbf{E}}\mathbf{t}_c(X,B)$. The induced functor*

$$\epsilon\colon \mathbf{Crys}(X,B) \to \acute{\mathbf{E}}\mathbf{t}_c(X,B)$$

is an equivalence of categories. It is compatible with all of the functors f^*, $\otimes$, $\otimes_B B'$, Sym^n, $\bigwedge^n$ and $R^i f_!$, and preserves flatness.

Remark 2. The category $\acute{\mathbf{E}}\mathbf{t}_c(X,B)$ possesses no duality. Therefore it can neither exist for $\mathbf{Crys}(X,B)$. Also only the functors f^*, $\otimes$ and $f_!$, which we have constructed on crystals, are well-behaved on $\acute{\mathbf{E}}\mathbf{t}_c(\ldots)$. Therefore one should not expect that all the functors $f^*, \otimes, f_!, f_*, \mathrm{Hom}, f^!$, postulated by Grothendieck for a good cohomological theory, exist for $\mathbf{Crys}(\ldots)$.

In special cases, one may construct some further functors. For instance in [Bö04] it is shown that for B finite or $B = A$ and any open immersion $j\colon U \hookrightarrow X$ there exists a meaningful functor

$$j_{\#}\colon \mathbf{Crys}(U, A) \longrightarrow \mathbf{Crys}(X, A) \tag{8}$$

in the sense of Definition 6. It corresponds to j_* in the étale theory. It is not called j_*, since in [BP04] this name was reserved for a different functor.

5 First applications

As explained in the introduction, one of the initial motivations to introduce the category of crystals was to give a cohomological proof of the rationality of L-functions of τ-sheaves. The rationality had been conjectured – at least in the case of families of Drinfeld modules – by Goss [Go91a], and a first proof had been given by Taguchi and Wan in [TW96] using analytical tools. Subsection 5.1 describes the cohomological proof from [BP04].

In addition to the rationality, the algebraic approach yields some extra information. Namely the L-function is expressible in terms of cohomology with compact support. These cohomology modules are in principle computable. The main result described in Subsection 5.2, makes crucial use of this.

Subsection 5.2 is concerned with a conjecture of Goss on analytic L-functions attached to families of Drinfeld modules. It asserts that these L-functions (with values in $\mathbb{C}_\infty$) have a meromorphic continuation to a function field analog of the complex plane, cf. [Go91a]. In the case $A = \mathbb{F}_q[t]$, this is the second main result in [TW96]. Later in [Bö02] Goss' conjecture was completely established for arbitrary A.

In the present section, we want to formulate the notions necessary to state the precise results and indicate some important steps in their proofs. Since for general A, Goss conjecture on meromorphy is rather technical to formulate, we will only do this in the case $A = \mathbb{F}_q[t]$. A detailed treatment of Subsection 5.1 can be found in [BP04], and of Subsection 5.2 in [Bö02].

5.1 *L*-functions of crystals

In this subsection we assume for simplicity of exposition that B is either a finite ring or a normal domain. We write $Q(B)$ for the ring of fractions of B, so that in the latter case $Q(B)$ is a field and in the former it is simply B again. All schemes will be of finite type over $\mathbb{F}_q$. By $|X|$ we denote the closed points of a scheme X. For $x \in |X|$ we denote its residue field by k_x and its degree by $d_x := [k_x : \mathbb{F}_q]$. Moreover $\underline{\mathcal{F}}$ will denote a flat B-crystal on X.

The aim is to explain how to attach L-functions to flat crystals on X, and state their main properties, i.e., the rationality, the invariance under $Rf_!$, and the invariance under change of coefficients.

Ultimately, these L-functions must be defined in terms of their underlying τ-sheaves, and at the same time invariant under nil-isomorphisms. For artinian B, we will use Theorem 9 to choose a good canonical representative.

When B is reduced, these L-functions satisfy all the usual cohomological formulas precisely (except duality). When B possesses non-zero nilpotent elements, however, these formulas hold only up to 'unipotent' factors. In some sense these factors correspond to nilpotent τ-sheaves and can therefore not be detected by our theory. So this defect is built in to our theory of crystals by its very construction.

As a preparation we briefly recall the theory of the dual characteristic polynomial for endomorphisms of projective modules. Suppose M is a finitely generated projective B-module and $\phi\colon M \to M$ is a B-linear endomorphism.

Lemma 1. *Let M' be any finitely generated projective B-module such that $M \oplus M'$ is free over B. Let $\phi'\colon M' \to M'$ be the zero endomorphism and t a new indeterminate.*

1. *The expression $\det_B\big(\mathrm{id} - t(\phi\oplus\phi') \bigm| M\oplus M'\big) \in 1 + tB[t]$ is independent of the choice of M'. It is called the* dual characteristic polynomial of (M,ϕ) *and denoted $\det_B(\mathrm{id} - t\phi \,|\, M)$.*
2. *The assignment $(M,\phi) \mapsto \det_B(\mathrm{id} - t\phi \bigm| M)$ is multiplicative in exact sequences.*

For any $x \in |X|$, the stalk $\underline{\mathcal{F}}_x$ is flat, and so is $\underline{\mathcal{F}}_x \otimes_B Q(B)$. By Theorem 9 the latter is canonically represented by the locally free τ-sheaf $(\underline{\mathcal{F}}_x \otimes_B Q(B))_{\mathrm{ss}}$. The endomorphism τ^{d_x} is $k_x \otimes Q(B)$-linear. By Lemma 1, part 1, the following definition makes sense.

Definition 11. *The* L-function *of $\underline{\mathcal{F}}$ at x is*

$$L(x,\underline{\mathcal{F}},t) := \det_{k_x\otimes Q(B)}\big(\mathrm{id} - t^{d_x}\tau^{d_x} \bigm| (\underline{\mathcal{F}}_x \otimes_B Q(B))_{\mathrm{ss}}\big)^{-1} \in k_x \otimes Q(B)[[t^{d_x}]].$$

Lemma 2. *1. The power series $L(x,\underline{\mathcal{F}},t)$ lies in $1 + t^{d_x}B[[t^{d_x}]]$.*
2. The assignment $\underline{\mathcal{F}} \mapsto L(x,\underline{\mathcal{F}},t)$ is multiplicative in short exact sequences.

The proof of part 1 needs our assumption on B.

As the number of points in $|X|$ of any given degree d_x is finite, we can form the product over the L-functions at all points $x \in |X|$ within $1 + tB[[t]]$.

Definition 12. *The* L-function *of $\underline{\mathcal{F}}$ is*

$$L(X,\underline{\mathcal{F}},t) \;:=\; \prod_{x\in|X|} L(x,\underline{\mathcal{F}},t) \;\in\; 1 + tB[[t]].$$

To state the main results, we need an equivalence relation on $1+tB[[t]]$:

Definition 13. *By $\mathfrak{n}_B$ we denote the nilradical of B, i.e., the ideal of B of nilpotent elements.*

For $P,Q \in 1+tB[[t]]$, we write $P \sim Q$ if and only if there exists $H \in 1+t\mathfrak{n}_B[t]$, such that $P = QH$.

If B is reduced, and so for instance if B is a normal domain, then $\mathfrak{n}_B = (0)$, and hence $P \sim Q$ is equivalent to $P = Q$.

Finally, if $h : B \to B'$ denotes a change of coefficients homomorphism, then its induced homomorphism $B[[t]] \to B'[[t]]$ is also denoted by h.

Theorem 12. *Suppose $f\colon Y \to X$ is any morphism between schemes of finite type, $h\colon B \to B'$ is any ring homomorphism and $\underline{\mathcal{G}}$ is a B-crystal of pullback type on Y. Then*

1. $L(Y,\underline{\mathcal{G}},t) \sim \prod_i L(X,R^i f_! \underline{\mathcal{G}},t)^{(-1)^i}$.
2. $L(Y,\underline{\mathcal{G}} \otimes_B B',t) \sim h(L(Y,\underline{\mathcal{G}},t))$, *with equality if B is artinian.*

Working in the context of derived categories, both parts can be proved more generally for any bounded complex $\underline{\mathcal{F}}^\bullet$ of flat crystals on Y, and with the complex $Rf_*\underline{\mathcal{F}}^\bullet$ instead of the crystals $R^i f_*\underline{\mathcal{F}}$. We alert the reader that the complex $Rf_*\underline{\mathcal{F}}^\bullet$ carries more information than the individual $R^i f_*\underline{\mathcal{F}}$.

A *rational function (over B)* is an element in $1+tB[[t]]$ of the form P/Q for suitable polynomials $P,Q \in 1+tB[t]$. The rationality of L-functions follows by applying Theorem 12, part 1, to the structure morphism $Y \to \operatorname{Spec} k$:

Corollary 1. *With Y, $\underline{\mathcal{G}}$ as in Theorem 12, the function $L(Y,\underline{\mathcal{G}},t)$ is rational.*

Proof (of Theorem 12 (sketch)). The proof of the two parts are independent. We shall omit the proof of of Part 2. Instead we indicate two alternative proofs of the trace formula in Part 1.

A *conventional proof* might go as follows, where first we assume B to be reduced: Standard fibering techniques reduce one to the proof in the case where $Y = \mathbb{A}^1$ and $X = \operatorname{Spec} \mathbb{F}_q$. The formula to be proved can then as in [SGA4$\frac{1}{2}$], ‘Rapport’ and ‘Fonction L mod ℓ^n et mod p’ be reduced to a trace formula over the symmetric powers of $\mathbb{A}^1_A$ for the corresponding exterior symmetric product of $\underline{\mathcal{G}}$. Again by induction on dimension, it suffices to prove this formula over $\mathbb{A}^1_A$. Its proof can be obtained from the Woodshole fixed point formula for coherent sheaves: see [SGA5] Exp. III Cor. 6.12.

Let now B be finite and non-reduced. The assertion can be reduced to affine Y over $\mathbb{F}_q$ and to τ-sheaves which are free over B on Y. Fixing a surjection $B' := \mathbb{F}_q[x_1,\ldots,x_k] \twoheadrightarrow B$, the τ-sheaf may be lifted to one over the reduced ring B'. For B' the trace formula has been proved already. Using Part 2, it follows for B.

The proof of Part 1, as given in [BP04] is significantly different and based on some ideas of Anderson, cf. [An00]. To sketch it, let us fix the following situation. Let $Y = \operatorname{Spec} R$ be smooth and affine over $X := \operatorname{Spec} \mathbb{F}_q$ of dimension e with f as the structure morphism, let B be a field, and suppose that $\underline{\mathcal{G}}$ is a locally free τ-sheaf.

Using coherent duality and the Cartier operator on $\omega_{Y/\mathbb{F}_q} := \bigwedge^e \Omega_{Y/\mathbb{F}_q}$, Anderson sets $\mathcal{G}^\vee := \mathcal{H}om(\mathcal{G}, \omega_{Y/\mathbb{F}_q})$ and obtains an $\mathcal{O}_Y \otimes B$-linear homomorphism $\kappa\colon \mathcal{G}^\vee \longrightarrow (\sigma \times \mathrm{id})^* \mathcal{G}^\vee$. Let M be the $R \otimes B$-module underlying $\mathcal{G}$ and $M^\vee$ that underlying $\mathcal{G}^\vee$. Anderson observes that κ is strongly contracting on $M^\vee$ in the following sense:

There exists a finite dimensional B-sub-vector space W of $M^\vee$ such that $\kappa(W) \subset W$ and such that $M^\vee = \bigcup_{i=1}^\infty \{m \in M^\vee \mid \kappa^i(m) \in W\}$. Such a subspace is termed a *nucleus* for $(M^\vee, \kappa)$. Using elementary means, Anderson proves that

$$L(Y, \underline{\mathcal{G}}, t) = \det_B(\mathrm{id} - t\kappa \mid W)^{(-1)^{e-1}}.$$

In [BP04] it is shown that for locally free $\mathcal{G}$ the only non-vanishing cohomology $R^i f_! \underline{\mathcal{G}}$ occurs in degree $i = e$, and that furthermore the dual of (W, κ) is via Serre duality nil-isomorphic to a suitable τ-sheaf representing $R^e f_! \underline{\mathcal{G}}$. Combining the above pieces, the proof is 'complete'.

The method just sketched has two main advantages. First, Anderson's proof is elementary. Second, our interpretation gives a cohomological interpretation for the nucleus and thus for his trace formula.

For finite B we have seen in the previous section that there is an equivalence of categories $\epsilon\colon \mathbf{Crys}(X, B) \longrightarrow \mathbf{\acute{E}t}_c(X, B)$ which preserves in particular the notion of flatness. As is well-known, to any constructible étale sheaf F of flat B-modules, one can also attach an L-function $L_{\text{ét}}(X, \mathsf{F}, t)$, e.g., [SGA4$\frac{1}{2}$], 'Rapport' and 'Fonction L mod ℓ^n'. In [BP04] the following is shown, which except for the comparison is also proved in [SGA4$\frac{1}{2}$]:

Theorem 13. *Suppose B is finite and $\underline{\mathcal{F}}$ is a flat B-crystal on X. Then*

$$L(X, \underline{\mathcal{F}}, t) = L_{\text{ét}}(X, \epsilon(\underline{\mathcal{F}}), t).$$

Hence $L_{\text{ét}}$ has a trace formula and is compatible with change of coefficients.

5.2 Goss' L-functions of crystals over A

To a scheme $\mathfrak{X}$ which is flat and of finite type over $\mathbb{Z}$, one can associate its ζ-function which is an analytic function that is convergent on a right half plane. A similar construction of global L-functions over function fields has been carried out by Goss for families of Drinfeld modules. It can easily be extended to τ-sheaves, and we now sketch this for $A = \mathbb{F}_q[t]$. For further details, for the more general case and for the case of v-adic L-functions, we refer the reader to [Go96], Chap. 8, and [Bö02].

Throughout this subsection, we fix a morphism $g\colon X \to \operatorname{Spec} A$ of finite type, and assume $B = A$. By $\underline{\mathcal{F}}$ we denote a flat A-crystal on X. The example considered originally was that of a Drinfeld A-module ϕ of rank r on $\mathcal{L}$ over some scheme X of finite type over $\mathbb{F}_q$. It yields a locally free τ-sheaf $\underline{\mathcal{M}}(\phi)$ $(= \underline{\mathcal{F}})$ of rank r, and for g one takes $\operatorname{char}_\phi\colon X \to \operatorname{Spec} A$.

Exponentiation of ideals

We begin by defining a substitute for the classically used expression p^{-s}, where $s \in \mathbb{C}$ and p is a prime number. Following Goss, one sets $S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$, which will replace the usual complex plane as the domain of L-functions. An element $s \in S_\infty$ will have components (z, w). One defines an addition by $(z_1, w_1) + (z_2, w_2) = (z_1 \cdot z_2, w_1 + w_2)$. As a uniformizer for $F_\infty \cong \mathbb{F}_q((1/t))$ we take $\pi_\infty := 1/t$.

Since any fractional ideal $\mathfrak{a}$ of A is principal, we may write it in the form (a) for some rational function $0 \neq a \in F$. The element $\langle a \rangle := a\pi_\infty^{-\operatorname{val}_\infty(a)}$ is a unit in $A_\infty \cong \mathbb{F}_q[[\pi_\infty]]$. We choose for a the *unique* generator of $\mathfrak{a}$ for which $\langle a \rangle$ is a 1-unit, and set $\langle \mathfrak{a} \rangle := \langle a \rangle$ as well as $\deg \mathfrak{a} := -\operatorname{val}_\infty(a)$. The exponentiation of ideals with elements in S_∞ is now defined as

Definition 14.

$$\{\textit{fractional ideals of } A\} \times S_\infty \to \mathbb{C}_\infty^* : (\mathfrak{a}, (z, w)) \mapsto \mathfrak{a}^s := z^{\deg \mathfrak{a}} \langle \mathfrak{a} \rangle^w.$$

The exponentiation is bilinear for multiplication on ideals, addition on S_∞ and multiplication on $\mathbb{C}_\infty^*$. Note that the exponentiation of any 1-unit with an element of $\mathbb{Z}_p$ is well-defined. The so-defined exponentiation *depends* on the choice of the uniformizing parameter $\pi_\infty = 1/t$.

One defines the embedding $\mathbb{Z} \hookrightarrow S_\infty : i \mapsto s_i := (\pi_\infty^{-i}, i)$, so that the element $\mathfrak{a}^{s_i} \in F$ is the unique generator of the ideal $\mathfrak{a}^i$ such that $\langle \mathfrak{a}^{s_i} \rangle$ is a 1-unit.

The definition of global *L*-functions

Let x be a closed point of X. Since X is of finite type over $\operatorname{Spec} A$, the point x lies above a unique closed point $\mathfrak{p} = \mathfrak{p}_x$ of $\operatorname{Spec} A$, and one has $d_\mathfrak{p} | d_x$ for their degrees over $\mathbb{F}_q$. Hence $L(x, \underline{\mathcal{F}}, t)^{-1} \in 1 + t^{d_x} B[t^{d_x}] \subset 1 + t^{d_\mathfrak{p}} B[t^{d_\mathfrak{p}}]$, and so $L(x, \underline{\mathcal{F}}, t)|_{t^{d_\mathfrak{p}} = \mathfrak{p}^{-s}}$ is well-defined. In [Bö02] or in many cases in [Go96], § 8, it is shown that there exists $c > 0$ such that the following Euler product converges for all $s \in \mathbb{H}_c := \{(z, w) \in \mathbb{C}_\infty^* \times \mathbb{Z}_p \mid |z| > c\}$:

Definition 15. *The* global L-function of $\underline{\mathcal{F}}$ *at* $s \in \mathbb{H}_c$ *is*

$$L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s) := \prod_{\mathfrak{p} \in \operatorname{Max}(A)} \prod_{\substack{x \in |X| \\ x \text{ above } \mathfrak{p}}} L(x, \underline{\mathcal{F}}, t)|_{t^{d_\mathfrak{p}} = \mathfrak{p}^{-s}} = \prod_{x \in |X|} L(x, \underline{\mathcal{F}}, t)|_{t^{d_{\mathfrak{p}_x}} = \mathfrak{p}_x^{-s}}.$$

Thus $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, \_\_)\colon \mathbb{H}_c \to \mathbb{C}_\infty$. The subset $\mathbb{H}_c \subset S_\infty$ is called a *half space (of convergence) of* S_∞ in analogy with the usual right half plane of $\mathbb{C}$.

Obviously, this definition depends on the morphism $g\colon X \to \operatorname{Spec} A$.

If $\underline{\mathcal{F}} = \underline{\mathcal{M}}(\phi)$ for some Drinfeld A-module ϕ on $\mathcal{L}$ over X, then Definition 15 agrees with the one originally given by Goss.

Meromorphy

For $c \in \mathbb{R}_{\geq 0}$ we define $D_c^* := \{z \in \mathbb{C}_\infty \mid |z| > c\} \subset \mathbb{P}^1(\mathbb{C}_\infty)$ as the *punctured 'open' disc around* ∞ *of radius* c, and $\bar{D}_c^* := \{z \in \mathbb{C}_\infty \mid |z| \geq c\}$ as the corresponding 'closed' disc. In particular $\mathbb{H}_c = D_c^* \times \mathbb{Z}_p$.

To give a precise meaning to 'entire', respectively 'meromorphic extension' of a global L-function to S_∞, one now changes ones view point. Namely, for any fixed $w \in \mathbb{Z}_p$ one regards an L-function as a function $D_c^* \to \mathbb{C}_\infty$. With respect to a suitable topology on the resulting functions $D_c^* \to \mathbb{C}_\infty$, the variation over $w \in \mathbb{Z}_p$ will be continuous. We now describe this topology:

For $D = \bar{D}_c^*$ and $c > 0$, or $D = D_c^*$ and $c \geq 0$, we define

$$\mathrm{C}^{\mathrm{an}}(D) := \Big\{ f = \sum_{n \geq 0} a_n z^{-n} \;\Big|\; a_n \in \mathbb{C}_\infty, f \text{ converges on } D \Big\}.$$

If $D = \bar{D}_c^*$, then $\mathrm{C}^{\mathrm{an}}(D)$ is isomorphic to the usual Tate algebra over $\mathbb{C}_\infty$, and one can define a Banach space structure on it by defining for any $f \in \mathrm{C}^{\mathrm{an}}(D)$ the norm $||f||_c := \sup_{z \in \bar{D}_c^*} |f(z)|$ (which is also multiplicative). In the case $D = D_c^*$ one can only obtain a Fréchet space. The procedure is slightly more involved. Namely, let $(c_n) \subset \mathbb{R}$ be a strictly decreasing sequence with limit c. For any two elements $f, g \in \mathrm{C}^{\mathrm{an}}(D)$ we define their distance as

$$\operatorname{dist}(f, g) := \sum_{m=1}^{\infty} 2^{-m} \frac{||f-g||_{c_m}}{1 + ||f-g||_{c_m}}.$$

One can show (a), that with respect to dist the $\mathbb{C}_\infty$-linear space $\mathrm{C}^{\mathrm{an}}(D)$ is a complete linear metric space, and (b), that the topology on this space does not depend on the choice of the sequence (c_n) (provided that $c_n \to c$ and $c_n > c$ for all n). In the following, $\mathbb{Z}_p$ is equipped with its usual locally compact topology.

Proposition 7. *If* $L(X, \underline{\mathcal{F}}, (z, w)))$ *converges on* $\mathbb{H}_c$*, then the function* $w \mapsto (z \mapsto L(X, \underline{\mathcal{F}}, (z, w)))$ *defines a continuous function* $\mathbb{Z}_p \to \mathrm{C}^{\mathrm{an}}(D_c^*)$.

This viewpoint bears some similarities to that of p-adic L-functions, where the complete, algebraically closed field $\mathbb{C}_p$ takes the role of $\mathrm{C}^{\mathrm{an}}(D_c^*)$.

For $c < c'$ one has the obvious inclusion $\mathrm{C}^{\mathrm{an}}(D_c^*) \subset \mathrm{C}^{\mathrm{an}}(D_{c'}^*)$, given as the identity on power series. Also since a power series is uniquely determined by its coefficients, an element in $\mathrm{C}^{\mathrm{an}}(D_{c'}^*)$ has at most one extension to an element in $\mathrm{C}^{\mathrm{an}}(D_c^*)$. Having this in mind, one introduces the following notions:

Definition 16 (Goss). *A continuous function* $\mathbb{Z}_p \to \mathrm{C}^{\mathrm{an}}(D_0^*)$ *is called* entire. *The quotient of two entire functions which are units on* D_c^* *for some* $c \gg 0$ *is called* meromorphic.

A global L-function $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)$ *is called* entire, *respectively* meromorphic on S_∞ *if there exists an entire, respectively meromorphic function h whose restriction* $h\colon \mathbb{Z}_p \to \mathrm{C}^{\mathrm{an}}(D_c^*)$ *agrees with* $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)$ *for* $c \gg 0$.

By the remark preceding the definition, there is at most one entire function h which extends a function $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)$. The same holds for meromorphic functions.

For a meromorphic function h, the values $h(i)$, $i \in \mathbb{Z}$, are its *special values*. In the examples of interest, the special values at $-\mathbb{N}_0$ will typically lie in $\mathbb{C}_\infty(z)$. Since both, $\mathbb{Z}$ and $\mathbb{N}_0$ are dense in $\mathbb{Z}_p$, the special values completely determine a meromorphic function. One has the following criterion in terms of special values for $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)$ to be entire.

Proposition 8. *Let* $\mathbb{H}_c$ *denote a half plane of convergence for* $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)$ *and write h for the corresponding continuous function* $\mathbb{Z}_p \to \mathrm{C}^{\mathrm{an}}(D_c^*)$. *Suppose there exists* $\varepsilon \in \{\pm 1\}$ *such that*

1. $h(i)^\varepsilon$ *is a polynomial in* z^{-1} *over* $\mathbb{C}_\infty$ *for all* $i \in -\mathbb{N}_0$, *and*
2. *the degrees of the polynomials* $h(i)^\varepsilon$, $i \in -\mathbb{N}_0$, *grow like* $\mathcal{O}(\log |i|)$.

Then $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)^\varepsilon$ *is entire.*

The assumptions of the theorem can typically be achieved for X smooth over $\mathbb{F}_q$ of dimension e, for $\underline{\mathcal{F}}$ locally free on $X \times \operatorname{Spec} A$, and $\varepsilon = (-1)^{e-1}$.

There are two ways to prove Proposition 8. The path taken in the proof of [Bö02], Theorem 4.15, uses directly p-adic interpolation properties. In [Go04a], Goss takes an alternative measure theoretic approach.

Special values

The *Carlitz* τ*-sheaf* $\underline{\mathcal{C}}$ over A on $\operatorname{Spec} A$ is the τ-sheaf corresponding to

$$(\mathbb{F}_q[t] \otimes \mathbb{F}_q[t], (1 \otimes t - t \otimes 1)(\sigma \otimes \mathrm{id})).$$

The following result, first observed by Taguchi and Wan, provides the key link between algebraic and analytic L-functions:

Proposition 9. *Suppose* $X = \operatorname{Spec} A \cong \mathbb{A}^1$ *and* $g\colon X \to \operatorname{Spec} A$ *is the identity. Then for* $i \in \mathbb{N}_0$ *one has*

$$L^{\mathrm{an}}(X, \underline{\mathcal{F}}, (z, 0) + s_{-i}) = L(X, \underline{\mathcal{F}} \otimes \underline{\mathcal{C}}^{\otimes i}, t)_{|t=z^{-1}},$$

where we recall $s\colon \mathbb{Z} \hookrightarrow S_\infty : i \mapsto s_i = (\pi_\infty^{-i}, i)$.

In light of Proposition 8, one would like to bound the degree of $L(X, \underline{\mathcal{F}} \otimes \underline{\mathcal{C}}^{\otimes i}, t)$ while i varies. For this we shall compute $Rg_!(\underline{\mathcal{F}} \otimes \underline{\mathcal{C}}^{\otimes i})$. If one has a τ-sheaf on $\operatorname{Spec} k$ representing it, its rank will bound the degree of the algebraic L-function. Let us denote by j the open immersion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ and $\bar{f}\colon \mathbb{P}^1 \to \operatorname{Spec} \mathbb{F}_q$ the structure morphism.

We proceed as follows: First one changes the coefficients from A to its fraction field F. Next, one extends the τ-sheaves $\underline{\mathcal{F}}$ and $\underline{\mathcal{C}}$ (over F) to τ-sheaves $\widetilde{\underline{\mathcal{F}}}$ and $\widetilde{\underline{\mathcal{C}}}$ on $\mathbb{P}^1$ which represent the crystals $j_!\underline{\mathcal{F}}$ and $j_!\underline{\mathcal{C}}$, respectively. Then, one computes $R^i\bar{f}_*(\widetilde{\underline{\mathcal{F}}} \otimes \widetilde{\underline{\mathcal{C}}}^{\otimes i})$ as a τ-sheaf over F on $\operatorname{Spec} \mathbb{F}_q$. In the case at hand, the τ-sheaves $R^i\bar{f}_*$, $i \neq 1$, will all be nilpotent. By the trace formula, Theorem 12, we thus have

$$L(X, \underline{\mathcal{F}} \otimes \underline{\mathcal{C}}^{\otimes i}, t) = \det_F(1 - t\tau \mid R^1\bar{f}_*\widetilde{\underline{\mathcal{F}}} \otimes \widetilde{\underline{\mathcal{C}}}^{\otimes i}).$$

Hence $L(X, \underline{\mathcal{F}} \otimes \underline{\mathcal{C}}^{\otimes i}, t)$ is a polynomial and its degree is bounded by the dimension of the F-vector space $H^1(\mathbb{P}^1_F, \widetilde{\mathcal{F}} \otimes \widetilde{\mathcal{C}}^{\otimes i})$.

The sheaf underlying $\widetilde{\underline{\mathcal{C}}}$ can be taken as $\mathcal{O}_{\mathbb{P}^1_F}(-2)$. If we follow the above recipe, then by the Riemann-Roch theorem, the dimension of $H^1(\mathbb{P}^1_F, \widetilde{\mathcal{F}} \otimes \widetilde{\mathcal{C}}^{\otimes i})$ will grow linearly in i. The degree of $L(X, \underline{\mathcal{F}} \otimes \underline{\mathcal{C}}^{\otimes i}, t) \in \mathbb{C}_\infty[t]$ can thus grow at most linearly. But we can do much better. Namely the sheaf $\underline{\mathcal{C}}^{\otimes p^i}$ is nil-isomorphic to $\underline{\mathcal{C}}^{(i)}$, which is defined by

$$(\mathbb{F}_q[t] \otimes \mathbb{F}_q[t], (1 \otimes t^{p^i} - t \otimes 1)(\sigma \otimes \mathrm{id})).$$

The latter (considered over F) has an extension $\widetilde{\underline{\mathcal{C}}}^{(i)}$ whose underlying sheaf is $\mathcal{O}_{\mathbb{P}^1_F}(-2)$. So if we write $i = a_0 + a_1 p + a_2 p^2 + \ldots$ in its p-adic expansion with $a_i \in \{0, 1, \ldots, p-1\}$, then

$$(\widetilde{\underline{\mathcal{C}}}^{(0)})^{\otimes a_0} \otimes (\widetilde{\underline{\mathcal{C}}}^{(1)})^{\otimes a_1} \otimes (\widetilde{\underline{\mathcal{C}}}^{(2)})^{\otimes a_2} \otimes \ldots$$

also represents the crystal $j_!\underline{\mathcal{C}}^i$, but its underlying sheaf is $\mathcal{O}_{\mathbb{P}^1_F}(-2(a_0 + a_1 + a_2 + \ldots))$. It follows that the degree of $L(X, \underline{\mathcal{F}} \otimes \underline{\mathcal{C}}^{\otimes i}, t)$ grows at most logarithmically in i. This combined with Props. 8 and 9 proves the following result which in the given form with a different proof is due to Taguchi and Wan, cf. [TW96]:

Theorem 14. *Suppose $X = \operatorname{Spec} A$, $g\colon X \to \operatorname{Spec} A$ is the identity, and $\underline{\mathcal{F}}$ is locally free over A on X. Then $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)$ is entire.*

Refining the above methods, in [Bö02] the following is shown:

Theorem 15. *Suppose X is Cohen-Macaulay and equidimensional of dimension e. Then for any locally free τ-sheaf $\underline{\mathcal{F}}$ over A on X the function $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)^{(-1)^{e-1}}$ is entire.*

Using the representability results for flat crystals, and the theory of iterates of characteristic polynomials, as developed in [BP04], one obtains, by decomposing X into a suitable finite union of regular locally closed subschemes, the following consequence:

Corollary 2. *Suppose* $X \to \operatorname{Spec} A$ *is a scheme of finite type and* $\underline{\mathcal{F}}$ *is a flat* A*-crystal on* X*. Then* $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)$ *is meromorphic.*

5.3 Open questions

While on the one hand side, we have seen that under some reasonable set of hypotheses the analytic functions $L^{\mathrm{an}}(X, \underline{\mathcal{F}}, s)$ are meromorphic, there remain many mysteries concerning these functions. The interpolation procedure seems to identify them as a kind of p-adic L-function interpolating special values at the negative integers. At the same time, these functions also have Euler products. We pose some open problems:

Question 1. What is the arithmetic meaning of the special values?

For an analog of the Riemann ζ-function, already in the work of Carlitz there appeared identities that are reminiscent of the formulas $\zeta(n) = \pi^n r_n$ for even $n \in \mathbb{N}$ and rational r_n. So Carlitz' formulas have some arithmetic meaning. For more on this, we refer to [Go96], § 8.18.

Question 2. Is there a conjecture à la Birch and Swinnerton-Dyer (BSD) for A-motives?

A naive analog of BSD can not hold, since it is known due to a result of Poonen, cf. [Po95], that the naive analog of the Mordell-Weil group for a Drinfeld A-module over a field L as in Section 2 is of infinite A-rank. In [An96] in certain cases, a finite rank A-module has been constructed, that could serve as a starting point to investigate such a conjecture.

Question 3. Is there a Riemann hypothesis or a (substitute for a) functional equation?

There is no duality to be expected for crystals or τ-sheaves, as explained in Remark 2. Nevertheless there are very intriguing calculations along these lines which are very suggestive although definitive conjectures cannot now be made, cf. [Go00] and [Go04].

6 Motives for Drinfeld cusp forms

Using geometric means, Scholl in [Sch90] has constructed for the space of cusp forms (over $\mathbb{Q}$) for each fixed weight and level a motive in the sense of Grothendieck. In this section, we want to describe a similar construction for Drinfeld modular forms in the function field case. It attaches a motive in the sense of Anderson to each space of Drinfeld cusp forms of fixed weight and level. Again for the sake of exposition, we only consider the simplest case $A = \mathbb{F}_q[t]$. Details of this appear in [Bö04].

6.1 Moduli spaces

Let $\mathfrak{n}$ be a proper-nonzero ideal of A. In Subsection 4.2, and there in particular in Theorem 2, we recalled the definition and existence of a fine moduli space $\mathfrak{Y}^r(\mathfrak{n})$ for Drinfeld A-modules of rank r and characteristic prime to $\mathfrak{n}$ that carry a level $\mathfrak{n}$-structure. From now on, we only consider the case $r = 2$, and therefore omit the superscript r whenever $r = 2$. As in Subsection 4.2, by $\mathcal{M}(\mathfrak{n})$ we denote the τ-sheaf corresponding to the universal Drinfeld A-module on $\mathfrak{Y}(\mathfrak{n})$, and by $g_\mathfrak{n} : \mathfrak{Y}(\mathfrak{n}) \to \operatorname{Spec} A(\mathfrak{n})$ its characteristic.

The first observation we will need in the following is due to Drinfeld:

Theorem 16. *The morphism $g_\mathfrak{n}$ has a (canonical) smooth compactification*

$$\begin{array}{ccc} \mathfrak{Y}(\mathfrak{n}) & \xhookrightarrow{\;j_\mathfrak{n}\;} & \mathfrak{X}(\mathfrak{n}) \\ & {\scriptstyle g_\mathfrak{n}}\searrow \quad \swarrow{\scriptstyle \bar{g}_\mathfrak{n}} & \\ & \operatorname{Spec} A(\mathfrak{n}). & \end{array}$$

The completion of $\mathfrak{X}(\mathfrak{n})$ along the complement $\mathfrak{X}(\mathfrak{n}) \setminus \mathfrak{Y}(\mathfrak{n})$ (considered as a reduced scheme) is (formally) smooth over $\operatorname{Spec} A(\mathfrak{n})$, and may be considered as a disjoint union of what one might call Drinfeld-Tate curves. They describe the degeneration of rank 2 to rank 1 Drinfeld modules and have properties analogous to the usual Tate curve. For details of this construction, cf. [Bö04], [vdH03], [Le01].

To describe the connectivity properties of $\mathfrak{Y}(\mathfrak{n})$ and the cusps, recall that in the case of elliptic curves the existence of the Weil-pairing yields a morphism of the corresponding (compactified) moduli space to $\operatorname{Spec} \mathbb{Z}[\zeta_N, \frac{1}{N}]$. Over this base the moduli space is geometrically connected, and so not over $\mathbb{Z}$ itself.

Similarly one has a pairing on rank 2 Drinfeld A-modules. It induces a smooth morphism $w_\mathfrak{n} : \mathfrak{Y}(\mathfrak{n}) \to \mathfrak{Y}^1(\mathfrak{n})$ to the moduli space of rank 1-Drinfeld modules with a level $\mathfrak{n}$-structure, which may be extended to a smooth proper morphism $\bar{w}_\mathfrak{n} : \mathfrak{X}(\mathfrak{n}) \to \mathfrak{Y}^1(\mathfrak{n})$, cf. [vdH03]. The situation together with the canonical morphism $\mathfrak{Y}^1(\mathfrak{n}) \to \operatorname{Spec} A(\mathfrak{n})$ is displayed in:

$$\begin{array}{ccccc} \mathfrak{Y}(\mathfrak{n}) & \hookrightarrow & \mathfrak{X}(\mathfrak{n}) & \hookleftarrow & \mathfrak{X}(\mathfrak{n}) \setminus \mathfrak{Y}(\mathfrak{n}) \\ & {\scriptstyle w_\mathfrak{n}}\searrow & \downarrow{\scriptstyle \overline{w}_\mathfrak{n}} & \swarrow{\scriptstyle \partial w_\mathfrak{n}} & \\ & & \mathfrak{Y}^1(\mathfrak{n}) & & \\ & & \downarrow{\scriptstyle g^1_\mathfrak{n}} & & \\ & & \operatorname{Spec} A(\mathfrak{n}). & & \end{array} \tag{9}$$

The morphism $g^1_{\mathfrak{n}}$ is finite étale, say of degree $d(\mathfrak{n})$; the morphism w_n is geometrically connected. Thus if we pass from $A(\mathfrak{n})$ to $\mathbb{C}_\infty$, the space $\mathfrak{Y}^1(\mathfrak{n})$ will decompose into $d(\mathfrak{n})$ copies of $\mathbb{C}_\infty$. Correspondingly, $\mathfrak{X}(\mathfrak{n})$ breaks up into $d(\mathfrak{n})$ components. Finally, under $\partial w_{\mathfrak{n}}$, the scheme $\mathfrak{X}(\mathfrak{n}) \setminus \mathfrak{Y}(\mathfrak{n})$ is isomorphic to a disjoint union of copies of $\mathfrak{Y}^1(\mathfrak{n})$. Their number is denoted by $c(\mathfrak{n})$.

6.2 Rigid analytic uniformization

Over $\mathbb{Z}[1/n]$, one has a compactification similar to $\mathfrak{X}(\mathfrak{n}) \to \operatorname{Spec} A(\mathfrak{n})$ for the arithmetic surfaces that arise as the moduli space of elliptic curves with a level N-structure. This is described in detail in [KM85]. If instead, one works over the complex numbers and the finer complex topology, the situation becomes considerably simpler. The resulting curves admit a uniformization by the upper half plane, and can be realized as quotients by congruence subgroups.

The analogous procedure in the function field setting is to base change $\mathfrak{Y}(\mathfrak{n})$ via $\iota\colon A(\mathfrak{n}) \hookrightarrow \mathbb{C}_\infty$ to a curve over $\operatorname{Spec} \mathbb{C}_\infty$. Now one regards the curve over $\mathbb{C}_\infty$ as a rigid analytic space – we write $\mathfrak{Y}(\mathfrak{n})^{\mathrm{rig}}$ –, which again yields a finer (Grothendieck) topology than the Zariski topology. For details on the rigidification functor $X \mapsto X^{\mathrm{rig}}$, we refer the reader to [BGR84].

As observed by Drinfeld, there is an analog of the upper half plane, usually denoted Ω. The rigidified moduli space $\mathfrak{Y}(\mathfrak{n})^{\mathrm{rig}}$ is in fact isomorphic to $\coprod_{i=1}^{d(\mathfrak{n})} \Gamma(\mathfrak{n})\backslash\Omega$ for a suitable quotient of Ω, which we now describe:

The points of Ω over $\mathbb{C}_\infty$ are given as $\Omega(\mathbb{C}_\infty) := \mathbb{P}^1(\mathbb{C}_\infty) \setminus \mathbb{P}^1(F_\infty)$. They are acted on by the group $\mathrm{GL}_2(F_\infty)$ via

$$\mathrm{GL}_2(F_\infty) \times \Omega(\mathbb{C}_\infty) \longrightarrow \Omega(\mathbb{C}_\infty) : \Big(\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big), z\Big) \mapsto \frac{az+b}{cz+d}.$$

To give Ω the structure of a rigid analytic space, we describe an admissible cover (which is the analog in rigid geometry of an atlas in differential geometry). Its construction can be best understood, if one introduces the reduction map of $\Omega(\mathbb{C}_\infty)$ to the corresponding Bruhat-Tits tree. The atlas we describe arises via pullback from a simple combinatorial Čech covering of the tree. Not wanting to introduce these notions, we now directly define the cover: Let $\mathbb{F}_\infty$ be the residue field of F_∞ and q_∞ its cardinality, and define

$$\mathfrak{U}_0(\mathbb{C}_\infty) := \Big\{ z \in \mathbb{C}_\infty \;\big|\; |z-\beta|_\infty \geq q_\infty^{-2/3} \text{ for all } \beta \in \mathbb{F}_\infty \text{ and } |z|_\infty \leq q_\infty^{2/3} \Big\}.$$

The set $\mathfrak{U}_0(\mathbb{C}_\infty)$ is rigid analytically equivalent to the unit disc with q_∞ smaller discs removed. One can show that

1. If for $\gamma \in \mathrm{GL}_2(F_\infty)$ the intersection $\mathfrak{U}_0(\mathbb{C}_\infty) \cap \gamma(\mathfrak{U}_0(\mathbb{C}_\infty))$ is non-empty, then one has precisely $q_\infty + 2$ possibilities: Either the intersection is $\mathfrak{U}_0(\mathbb{C}_\infty)$, or it is

$$\mathfrak{U}_1(\mathbb{C}_\infty) = \Big\{ z \in \mathbb{C}_\infty \;\big|\; q_\infty^{1/3} \leq |z|_\infty \leq q_\infty^{2/3} \Big\},$$

or there exists some $\beta \in \mathbb{F}_\infty$ such that it is of the form

$$\left\{ z \in \mathbb{C}_\infty \mid q_\infty^{-1/3} \geq |z - \beta|_\infty \geq q_\infty^{-2/3} \right\}.$$

2. The sets in 1 different from $\mathfrak{U}_0(\mathbb{C}_\infty)$ are all translates of $\mathfrak{U}_1(\mathbb{C}_\infty)$.
3. The sets $\gamma\,\mathfrak{U}_0(\mathbb{C}_\infty)$, $\gamma \in \mathrm{GL}_2(F_\infty)$, form an admissible covering of $\Omega(\mathbb{C}_\infty)$.

To define a geometry (in this case a rigid analytic structure), one also has to describe a set of functions on the atlas given. A function $f\colon \mathfrak{U}_0(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ is *rigid analytic on* $\mathfrak{U}_0(\mathbb{C}_\infty)$, if and only if it can be written as a series

$$\sum_{n \in \mathbb{N}_0} a_n z^n + \sum_{\beta \in \mathbb{F}_\infty} \sum_{n \in \mathbb{N}} b_{n,\beta}(z - \beta)^{-n}$$

which converges on all of $\mathfrak{U}_0(\mathbb{C}_\infty)$. The latter simply means that the sequences $(|a_n|q^{2/3n})$ and $(|b_{n,\beta}|q^{-2/3n})$, for all $\beta \in \mathbb{F}_\infty$, tend to zero.

Definition 17. *A function* $f : \Omega(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ *is* rigid analytic on $\Omega(\mathbb{C}_\infty)$*, if for all* $\gamma \in \mathrm{GL}_2(F_\infty)$ *the restriction of* $f \circ \gamma$ *to* $\mathfrak{U}_0(\mathbb{C}_\infty)$ *is rigid analytic.*

To describe $\Gamma(\mathfrak{n})\backslash\Omega$, we recall that one defines

$$\Gamma(\mathfrak{n}) := \{\gamma \in \mathrm{GL}_2(A) \mid \gamma \equiv \mathrm{id} \pmod{\mathfrak{n}}\}.$$

It is a discrete subgroup of $\mathrm{GL}_2(F_\infty)$, and thus acts on $\Omega(\mathbb{C}_\infty)$. Say we fix $\gamma \in \mathrm{GL}_2(F_\infty)$ and abbreviate $\mathfrak{U} := \gamma\mathfrak{U}_0$. Then for $\gamma_0 \in \Gamma(\mathfrak{n})$ one either has $\gamma_0\mathfrak{U} = \mathfrak{U}$ or $\gamma_0\mathfrak{U} \cap \mathfrak{U} = \emptyset$. The former case only occurs a finite number of times, so that the stabilizer $\mathrm{Stab}_{\Gamma(\mathfrak{n})}(\mathfrak{U})$ of $\mathfrak{U}$ in $\Gamma(\mathfrak{n})$ is finite. One may define rigid analytic quotients $\mathrm{Stab}_{\Gamma(\mathfrak{n})}(\mathfrak{U})\backslash\mathfrak{U}$, and these can be glued to define a rigid space $\Gamma(\mathfrak{n})\backslash\Omega$.

6.3 Cusp forms

Following the case of elliptic modular forms over number fields, one can define Drinfeld modular functions (and cusp forms) over function fields as follows:

Definition 18. *A rigid analytic function* $f : \Omega(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ *is called a* modular function *of weight* $k \in \mathbb{N}$ *for* $\Gamma(\mathfrak{n})$*, if it satisfies the identity*

$$f(\gamma z) = (cz + d)^k f(z) \quad \textit{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathfrak{n}) \textit{ and } z \in \Omega(\mathbb{C}_\infty).$$

The modular functions of level $\mathfrak{n}$ are invariant under the operation of

$$\Gamma_\infty(\mathfrak{n}) := \left\{ \gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathfrak{n} \right\},$$

i.e., under translation by all $b \in \mathfrak{n}$. Another such function is

$$e_{\mathfrak{n}}(z) := z \prod_{\lambda \in \mathfrak{n} \setminus \{0\}} \Big(1 - \frac{z}{\lambda}\Big).$$

The function $t_{\mathfrak{n}} := e_{\mathfrak{n}}^{-1}$ is, in a suitable sense, a uniformizing parameter of $\mathfrak{n}$-invariant functions near ∞. Therefore any modular function has a Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n t_{\mathfrak{n}}^n(z)$$

convergent for $|z|_i \gg 0$, where $|z|_i := \inf\{|z - a| \mid a \in F_\infty\}$.

If f is a modular function of weight k for $\Gamma(\mathfrak{n})$, then so is $f \circ \gamma$ for all $\gamma \in \mathrm{GL}_2(A)$, because $\Gamma(\mathfrak{n})$ is normal in $\mathrm{GL}_2(A)$. Therefore one can equally consider the expansion $\sum_{n \in \mathbb{Z}} a_{n,\gamma} t_{\mathfrak{n}}^n(z)$ of $f \circ \gamma$. Since $\mathrm{GL}_2(A)$ acts transitively on the "cusps" of $\Gamma \backslash \Omega$, following the case of elliptic modular forms one defines:

Definition 19. *A modular function f of weight k for $\Gamma(\mathfrak{n})$ is called a*

$$\left\{\begin{array}{c}\text{modular form}\\ \text{cusp form}\\ \text{double cusp form}\end{array}\right\} \iff \forall\, \gamma \in \mathrm{GL}_2(A)\, \forall n \in \mathbb{Z} \text{ with } \left\{\begin{array}{c} n < 0\\ n < 1\\ n < 2\end{array}\right\} a_{n,\gamma} = 0.$$

By $M_k(\Gamma(\mathfrak{n}), \mathbb{C}_\infty)$, we denote the $\mathbb{C}_\infty$-vector space of modular forms for $\Gamma(\mathfrak{n})$ of weight k and by $S_k(\Gamma(\mathfrak{n}), \mathbb{C}_\infty)$ and $S_k^{\mathrm{dc}}(\Gamma(\mathfrak{n}), \mathbb{C}_\infty)$ its subspaces of cusp and double cusp forms.

The number of conditions imposed in Definition 19 is finite, since it suffices to require these condition for matrices γ which form a set of representatives of $\Gamma(\mathfrak{n}) \backslash \mathrm{GL}_2(A)$.

Elliptic double cusp forms are usually not considered, since the have no meaningful interpretation. One reason to introduce them in the Drinfeld modular setting is given below in Theorem 17.

To define modular forms on $\mathfrak{Y}(\mathfrak{n})^{\mathrm{rig}}$ one uses its identification with $\coprod_{i=1}^{d(\mathfrak{n})} \Gamma(\mathfrak{n}) \backslash \Omega$. Formally we set

$$M_k(\mathfrak{n}, \mathbb{C}_\infty) := M_k(\Gamma(\mathfrak{n}), \mathbb{C}_\infty)^{d(\mathfrak{n})}, \tag{10}$$

and similarly for cusp and double cusp forms.

As in the classical situation, one may define Hecke operators (e.g., as correspondences) which act on Drinfeld modular forms. These preserve the subspaces of cusp and double cusp forms. (Depending on their normalization, Hecke operators may also "permute" the components of $\mathfrak{Y}(\mathfrak{n})/\mathbb{C}_\infty$.)

We conclude this subsection with two examples. For the first, recall that $\mathfrak{Y}(\mathfrak{n})$ is an affine scheme, say equal to $\operatorname{Spec} R_{\mathfrak{n}}$ for some smooth $A(\mathfrak{n})$-algebra $R_{\mathfrak{n}}$. As remarked in Theorem 2, the universal rank 2 Drinfeld A-module is a homomorphism $A \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,\mathfrak{Y}(\mathfrak{n})})$, i.e., a homomorphism

$$\phi_{\mathfrak{n}} \colon A \to R_{\mathfrak{n}}\{\tau\}.$$

There are morphisms from $R_{\mathfrak{n}}$ to its rigidification, and then from the latter to the ring of global rigid analytic sections $\Gamma(\Omega, \mathcal{O}_{\Omega}^{\mathrm{rig}})^{d(\mathfrak{n})}$ of $\bigsqcup_{i=1}^{d(\mathfrak{n})} \Omega(\mathbb{C}_\infty)$. This yields a homomorphism

$$\phi(\mathfrak{n})\colon A \to \Gamma(\Omega, \mathcal{O}_{\Omega}^{\mathrm{rig}})^{d(\mathfrak{n})}\{\tau\}.$$

Proposition 10. *For any $a \in A$ consider $\phi(\mathfrak{n})_a$. The coefficient of its leading term $\tau^{2\deg a}$ lies in $S_{q^{2\deg a}-1}(\mathfrak{n}, \mathbb{C}_\infty)$, the coefficients of the remaining terms τ^i lie in $M_{q^i-1}(\mathfrak{n}, \mathbb{C}_\infty)$.*

As for elliptic modular forms one has an interpretation of the global sections of the sheaf of differentials on $\mathfrak{X}(\mathfrak{n})$ in terms of modular forms:

Theorem 17. *There is a canonical isomorphism*

$$H^0(\mathfrak{X}(\mathfrak{n})^{\mathrm{rig}}, \Omega_{\mathfrak{X}(\mathfrak{n})^{\mathrm{rig}}/\mathbb{C}_\infty}) \cong S_2^{\mathrm{dc}}(\mathfrak{n}, \mathbb{C}_\infty).$$

Without going into any details, the reason for the occurrence of double cusp forms is the following. Suppose $f(z)$ is a Drinfeld modular form of weight 2 for $\Gamma(\mathfrak{n})$. Then $f(z)\mathrm{d}z$ is a global $\Gamma(\mathfrak{n})$-invariant differential form on $\Omega(\mathbb{C}_\infty)$. To investigate its behavior near the cusp described by $t_{\mathfrak{n}}$, observe that one has $\mathrm{d}e_{\mathfrak{n}} = \mathrm{d}z$ from the definition of $e_{\mathfrak{n}}$, and hence

$$\mathrm{d}t_{\mathfrak{n}} = \mathrm{d}(e_{\mathfrak{n}}^{-1}) = -e_{\mathfrak{n}}^{-2}\mathrm{d}z = -t_{\mathfrak{n}}^2\mathrm{d}z.$$

Thus near the cusp for $t_{\mathfrak{n}}$, the function $f(z)\mathrm{d}z$ is a power series in $t_{\mathfrak{n}}$ times $-\frac{1}{t_{\mathfrak{n}}^2}\mathrm{d}t_{\mathfrak{n}}$. Hence if we start with a double cusp form, we obtain a global differential on $\mathfrak{X}(\mathfrak{n})$, and vice versa.

6.4 The motive

Following the guide by Eichler-Shimura and Deligne, we now define for any $k \geq 2$ the locally free τ-sheaf $\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)} := \mathrm{Sym}^{k-2}\,\underline{\mathcal{M}}(\mathfrak{n})$, and the A-crystal

$$\underline{\mathcal{S}}^{(k)}(\mathfrak{n}) := R^1 g_{\mathfrak{n}!}\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)} \tag{11}$$

on $\mathrm{Spec}\,A(\mathfrak{n})$. By Proposition 3 the τ-sheaf $\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)}$ is locally free of rank $k-1$. Because $\mathfrak{Y}(\mathfrak{n})$ is affine, the corresponding module is projective and finitely generated. If we extend it to a free module and choose $\tau = 0$ on the complement, we find that $\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)}$ is of pullback type. Also it is not difficult to see that the crystal $R^0 g_{\mathfrak{n}!}\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)}$ is zero. Since moreover $\bar{g}_{\mathfrak{n}}$ is smooth and proper of relative dimension 1, Proposition 5 yields:

Proposition 11. *The crystal $R^i g_{\mathfrak{n}!}\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)}$ is zero for $i \neq 1$. The crystal $\underline{\mathcal{S}}^{(k)}(\mathfrak{n}) = R^1 g_{\mathfrak{n}!}\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)}$ is of pullback type and hence flat.*

In [Bö04], jointly with R. Pink we computed some explicit examples of such motives for $A = \mathbb{F}_q[t]$ and $\mathfrak{n} = (t)$. Other explicit examples are given in Corollaries 4, 5 and 6.

Considering geometric correspondences, one can define Hecke operators T_v for all places of A prime to $\mathfrak{n}$. They naturally act on the crystals $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})$. Let $\mathfrak{p}_v$ be the maximal ideal corresponding to v. The Hecke operators will still act as Hecke operators on the reduction of $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})$ to the fiber at v, i.e., its pullback along $\operatorname{Spec} A/\mathfrak{p}_v \hookrightarrow \operatorname{Spec} A(\mathfrak{n})$. As basically already observed by Drinfeld, one has an Eichler-Shimura relation:

Theorem 18. *The action of T_v on the fiber of $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})$ at v is given by the action of the geometric Frobenius* Frob_v *at v on this fiber.*

6.5 Its analytic realization

We now follow the guide of A-motives to define an analytic realization of the crystals $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})$. First we pull back this crystal via $A(\mathfrak{n}) \hookrightarrow \mathbb{C}_\infty$ to a crystal over A on $\mathbb{C}_\infty$. Because it is flat, we may by using Proposition 6 represent it by a τ-module $(S^{(k)}(\mathfrak{n}), \tau)$ over A on $\mathbb{C}_\infty$ whose underlying sheaf is finitely generated projective, and on which τ is injective. In [Bö04] it is shown:

Theorem 19. *The τ-module $(S^{(k)}(\mathfrak{n}), \tau)$ is uniformizable in the sense of Definition 3.*

Therefore we call

$$\left(S^{(k)}(\mathfrak{n})\langle t\rangle\right)^{\tau}$$

the *analytic realization of the motive* $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})$. This realization carries an induced Hecke action. The following main result is shown in [Bö04]:

Theorem 20. *There is a canonical Hecke-equivariant isomorphism*

$$\mathrm{Hom}_A\left(\left(S^{(k)}(\mathfrak{n})\langle t\rangle\right)^{\tau}, \Omega_A\right) \otimes_A \mathbb{C}_\infty \cong S_k(\mathfrak{n}, \mathbb{C}_\infty).$$

The result should be compared with formula 3. The proof uses rigid analytic tools and an explicit combinatorial Čech covering of $\mathfrak{X}(\mathfrak{n})^{\mathrm{rig}}$. It would go beyond the scope of this article to give details.

Using the Hecke action one may define a Hecke-invariant filtration on $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})$ whose subquotients are flat crystals $\underline{\mathcal{S}}_f$ corresponding to (generalized) cuspidal Drinfeld Hecke eigenforms f. Neither the filtration, nor the crystal $\underline{\mathcal{S}}_f$ are canonical, and more precisely the crystal $\underline{\mathcal{S}}_f$ corresponds to the Galois orbit of f, and one has to be aware that the Hecke action on $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})$ may not be semi-simple. Nevertheless, these subquotients are useful.

6.6 Its étale realizations

Let us fix a place v of A. Then $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})\otimes_A A/\mathfrak{p}_v^n$ defines a flat crystal over $A/\mathfrak{p}_v^n$ on $\operatorname{Spec} A(\mathfrak{n})$. Via the functor ϵ from Subsection 4.8, we obtain a lisse étale sheaf of $A/\mathfrak{p}_v^n$-modules on $\operatorname{Spec} A(\mathfrak{n})$, which say we denote by $S^{\text{ét}}(\mathfrak{n}, A/\mathfrak{p}_v^n)$. Varying n, these sheaves form a compatible system, and thus a v-adic étale sheaf on $\operatorname{Spec} A(\mathfrak{n})$. Since all sheaves in this systems are lisse, we obtain a Galois representation

$$\rho_{\mathfrak{n},k,v}\colon G_F \to \mathrm{GL}_{d_{\mathfrak{n},k}}(A_v),$$

where $d_{\mathfrak{n},k}$ is the dimension of $S_k(\mathfrak{n}, \mathbb{C}_\infty)$.

The filtration on $\underline{\mathcal{S}}^{(k)}(\mathfrak{n})$ described at the end of the previous subsection induces also a filtration on the compatible system $S^{\text{ét}}(\mathfrak{n}, A/\mathfrak{p}_v^n)$. The subquotients yield the compatible systems $\epsilon(\underline{\mathcal{S}}_f \otimes_A A/\mathfrak{p}_v^n)$ corresponding to (Galois orbits of generalized) cuspidal Drinfeld Hecke eigenforms f. The correspondence can be made precise by using the Eichler-Shimura relation from Theorem 18. One obtains:

Theorem 21. *Let f be a cuspidal Drinfeld Hecke eigenform (over $\mathbb{C}_\infty$) and denote by F_f the field generated over F by the Hecke eigenvalues a_w of f where w runs through all places of A prime to $\mathfrak{n}$. Then $[F_f : F]$ is finite and for any place v of A there exists a place v' of F_f above v and a representation*

$$\rho_{f,v}\colon \mathrm{Gal}(F^{\text{sep}}/F) \longrightarrow \mathrm{GL}_1((F_f)_{v'})$$

uniquely characterized by

$$\rho_{f,v}(\mathrm{Frob}_w) = a_w \tag{12}$$

for all places w of A which are prime to $\mathfrak{n}\mathfrak{p}_v$.

This result is strikingly different from the analogous one for elliptic modular forms since there the representations are 2-dimensional. To explain this, we recall the following observation on Hecke operators which dates back to Gekeler and Goss. Namely one can define Hecke operator $T_\mathfrak{m}$ for any non-zero ideal $\mathfrak{m}$ prime to $\mathfrak{n}$. In characteristic p they satisfy $T_{\mathfrak{m}\mathfrak{m}'} = T_\mathfrak{m} T_{\mathfrak{m}'}$ for any ideals $\mathfrak{m}$, $\mathfrak{m}'$, and in particular for $\mathfrak{m}$ a power of some prime ideal $\mathfrak{p}$. This is different from the case of characteristic zero, *but* it simply follows from the usual relation by reduction modulo p. Another reason why one should expect abelian representations is given below Corollary 6.

The characterizing property (12) is basically the same at all places v of A. This means that the representations $\rho_{f,v}$ form a compatible system of v-adic abelian representations of G_F. (The same holds for the semisimplification of the representations $\rho_{\mathfrak{n},k,v}$.) Thus extending the results of [Kh04] to the function field case, it seems natural to expect that to any Drinfeld cusp form one can attach a Grössencharacter χ_f of type A_0 such that the compatible family $\rho_{f,v}$ arises from χ_f in the way described in [Kh03], § 4, and [Go92]. Therefore the following natural question arises:

Question 4. Which Grössencharacters of F of type A_0 do arise from Drinfeld modular forms?

Can any Grössencharacter of F of type A_0 be twisted by a power of the Grössencharacter arising from the Carlitz-module (as in [Go92]), such that it arises from a Drinfeld modular form?

Recall that $\underline{\mathcal{C}}$ is the Carlitz-module defined above Proposition 9. Then Question 4 is a generalization of the following problem raised in [Go02]:

Question 5. Can one find for any Drinfeld $\mathbb{F}_q[t]$-module ϕ on $\operatorname{Spec}\mathbb{F}_q[t]$ of rank 1 an $n \in \mathbb{N}_0$ such that $\underline{\mathcal{M}}(\phi) \otimes \underline{\mathcal{C}}^{\otimes n}$ is the motive of a modular form?

Namely any such Drinfeld module determines a Hecke-character of type A_0 and is moreover uniquely determined by this Hecke-character.

6.7 *L*-functions

Having the (non-canonically defined) crystal $\underline{\mathcal{S}}_f$ attached to any cuspidal Drinfeld Hecke eigenform f of level $\mathfrak{n}$, using the formalism described in Subsection 5.2 one can attach an analytic L-function to it. It is independent of the choice of $\underline{\mathcal{S}}_f$. This yields non-trivial factors in the Euler product at all primes not dividing $\mathfrak{n}$. However the function f may be an old form, i.e., defined over some smaller level $\mathfrak{n}'$. Thus it would be desirable to also have Euler factors at primes in $\operatorname{Spec} A/\mathfrak{n} \setminus \operatorname{Spec} A/\mathfrak{n}'$. One way to achieve this is to assign to f the analytic L-function of the maximal model of $\underline{\mathcal{S}}_f$ in the sense of Gardeyn, cf. Definition 6. This assignment is now also independent of the level in which f was found. Using in particular Theorem 14 and Theorem 21, one shows:

Corollary 3. *For f a cuspidal Drinfeld Hecke eigenform of minimal level $\mathfrak{n}$ and with Hecke eigenvalues a_v, one has*

$$L_f^{\mathrm{an}}(s) = \prod_{v \in \operatorname{Max}(A(\mathfrak{n}))} \Big(1 - \frac{a_v}{\mathfrak{p}^s}\Big)^{-1}$$

for $s = (z, w) \in S_\infty$ with $|z| \gg 0$. The function $L_f^{\mathrm{an}}(s)$ is entire in the sense of Definition 16.

In [Go91] two further analytic L-functions are attached to a cuspidal Drinfeld modular form. These are known to be different from that in Corollary 3.

Question 6. What is the relation between these L-functions if any?

The assignment $f \mapsto L_f^{\mathrm{an}}(s)$ described in Corollary 3, attaches to a Hecke eigenform an L-function. However unlike in the situation for elliptic modular forms, there is no normalization for such forms in the Drinfeld modular setting. In the elliptic modular setting the typical requirement is that the first Fourier coefficient of f is 1. As the examples of doubly cuspidal Drinfeld Hecke eigenforms show, this coefficient may be zero in our setting.

Question 7 (Goss). Is there a canonical normalization of a cuspidal Drinfeld Hecke eigenform?

If the answer would be yes, then by superposition one could attach an L-function to any Drinfeld cusp form.

Question 8. Can the assignment $f \mapsto L_f(s)$ be realized by an analog of the usual Mellin transform?

6.8 Double cusp forms

We saw in Theorem 17, that double cusp forms do play an important role in the theory of Drinfeld modular forms. So one may wonder whether there is also a motive describing these. The answer, in short, is yes, and we will explain some of it, since it leads to another interesting question.

Let $k \geq 2$. In Subsection 6.4, we defined $\underline{\mathcal{S}}^{(k)}(\mathfrak{n}) = R^1\bar{g}_\mathfrak{n}\, j_{\mathfrak{n}!}\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)}$, i.e., we first extended the crystal $\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)}$ to $\mathfrak{X}(\mathfrak{n})$ by zero, and then we computed its first cohomology. In Remark 2, we noted that Gardeyn's notion of maximal model leads to a functor $j_\#$ on A-crystals provided the base X was of finite type over some field. Therefore we may define

$$\underline{\mathcal{S}}^{(k)}_{\mathrm{dc}}(\mathfrak{n}) = R^1\bar{g}_\mathfrak{n}\, j_{\mathfrak{n}\#}\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)}.$$

One can now formulate (by adding the subscript $_{\mathrm{dc}}$) and prove the precise analog of Theorems 19 and 20 for double cusp forms, cf. [Bö04].

It is in fact possible to completely determine the discrepancy between cusp and double cusp forms. Namely for any $k \geq 2$ one has a short exact sequence

$$0 \longrightarrow j_!\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)} \longrightarrow j_\#\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)} \longrightarrow \underline{\mathbb{1}}_{\mathfrak{X}(\mathfrak{n})\setminus\mathfrak{Y}(\mathfrak{n}),A} \longrightarrow 0$$

of crystals. The long exact sequence of cohomology then yields a 4-term exact sequence of crystals

$$0 \to \bar{g}_{\mathfrak{n}*}j_\#\underline{\mathcal{M}}(\mathfrak{n})^{(k-2)} \to \bar{g}_{\mathfrak{n}*}\underline{\mathbb{1}}_{\mathfrak{X}(\mathfrak{n})\setminus\mathfrak{Y}(\mathfrak{n}),A} \to \underline{\mathcal{S}}^{(k)}(\mathfrak{n}) \to \underline{\mathcal{S}}^{(k)}_{\mathrm{dc}}(\mathfrak{n}) \to 0. \qquad (13)$$

The properties of diagram (9) yield $\bar{g}_{\mathfrak{n}*}\underline{\mathbb{1}}_{\mathfrak{X}(\mathfrak{n})\setminus\mathfrak{Y}(\mathfrak{n}),A} \cong (g^1_{\mathfrak{n}*}\underline{\mathbb{1}}_{\mathfrak{Y}^1(\mathfrak{n}),A})^{c(\mathfrak{n})}$. Moreover for $k \neq 2$ the left hand term vanishes, for $k = 2$ it is isomorphic to $g^1_{\mathfrak{n}*}\underline{\mathbb{1}}_{\mathfrak{Y}^1(\mathfrak{n}),A}$.

We define $\overline{S}_k(\mathfrak{n}, \mathbb{C}_\infty) := S_k(\mathfrak{n}, \mathbb{C}_\infty)/S^{\mathrm{dc}}_k(\mathfrak{n}, \mathbb{C}_\infty)$, and set $\delta_k := 0$ for $k \geq 3$ and $\delta_k := 1$ for $k = 2$. Sequence (13) with the above identifications, the duality in Theorem 20, and the analogous duality for double cusp forms, prove:

Corollary 4. *There is a fixed Hecke-module of dimension $d(\mathfrak{n})$ depending on $\mathfrak{n}$ but not on k, such that $\overline{S}_k(\mathfrak{n}, \mathbb{C}_\infty)$ is the direct sum of $c(\mathfrak{n}) - \delta_k$ copies of it.*

The Hecke-module in question arises from the arithmetic of $\mathfrak{Y}^1(\mathfrak{n})$.

Question 9. Can one give an explicit basis for the cuspidal Drinfeld Hecke eigenforms which are not double cusp forms, in a way similarly explicit to the the description one has for Eisenstein series?

It is equally interesting to consider the consequences of (13) for the étale realization of our motives. With δ_k as above, one obtains:

Corollary 5. *The v-adic étale realization corresponding to $\overline{S}_k(\mathfrak{n},\mathbb{C}_\infty)$ consists of $(c(\mathfrak{n})-\delta_k)$ copies of $H^1(\mathfrak{Y}(\mathfrak{n})^1/F^{\mathrm{sep}},\mathbb{F}_q)\otimes F_v$, considered as a Galois representation of G_F (which is unramified outside $\mathfrak{n}$).*

In particular this shows that the étale realizations corresponding to the Hecke eigenforms in $\overline{S}_k(\mathfrak{n},\mathbb{C}_\infty)$ give rise to Galois representations over $\mathbb{F}_q$, and thus with finite image. The proof of the corollary uses, among other things, the compatibility of the functor ϵ of Theorem 11 with coefficient change and with proper push-forward.

Specializing (13) to $k=2$, and passing to étale realizations one finds:

Corollary 6. *The v-adic étale realization of $\underline{\mathcal{S}}_{\mathrm{dc}}^{(2)}(\mathfrak{n})$ is the Galois representation given of G_F on $H^1(\mathfrak{X}(\mathfrak{n})/F^{\mathrm{sep}},\mathbb{F}_q)\otimes F_v$.*

Again this describes Galois representations over $\mathbb{F}_q$ and thus with finite image. It is known that the curve $\mathfrak{X}(\mathfrak{n})/F$ is ordinary, and hence so is its Jacobian, which, say, we denote by $J(\mathfrak{n})$. One therefore has

$$H^1(\mathfrak{X}(\mathfrak{n})/F^{\mathrm{sep}},\mathbb{F}_q)\cong J(\mathfrak{n})[p](F^{\mathrm{sep}})\cong \mathbb{Z}/(p)^{\dim J(\mathfrak{n})},$$

where the first isomorphism is an isomorphism of Galois modules. Since in particular the semisimplification of the module in the middle is abelian, this gives another indication for the abelianess of the representations $\rho_{f,v}$.

The examples in [Bö04] show that the image of $\rho_{f,v}$ is typically infinite for $f\in S_k^{\mathrm{dc}}(\mathfrak{n},\mathbb{C}_\infty)$ and $k>2$. This is related to a notion of weight that one can attach to (pure) motives. Again this notion goes back to Anderson. Its definition is similar to the notion of weight for ℓ-adic sheaves due to Deligne. Now for an elliptic cuspidal Hecke eigenform of weight k one knows that the weight of its ℓ-adic Galois representation is $k-1$, which in turn yields the Ramanujan-Peterson conjecture. By considering examples we expect that for cuspidal Drinfeld Hecke eigenforms of weight k the following holds: The weight of their v-Galois representation is well-defined and an integer in $[0,\ldots,\frac{k-1}{2}]$. A proof of this is still lacking.

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